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AN ANALYTICAL STUDY OF SIMPLE FRAMES CONSIDERING THE SECONDARY EFFECT OF BENDING DUE TO AXIAL LOADS

A THESIS

Presented to

The Faculty of the Graduate Division

by

Clifford O. Hays, Jr.

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Civil Engineering

Georgia Institute of Technology

August, 1966

AN ANALYTICAL STUDY OF SIMPLE FRAMES CONSIDERING THE

SECONDARY EFFECT OF BENDING DUE TO AXIAL LOADS

Approved:

Chairman

Date approved by Chairman: 31 May 1969

DEDICATION

This thesis is dedicated to my beloved wife, Joyce, for her countless sacrifices during the past year.

ACKNOWLEDGMENTS

The author would like to express his deepest appreciation to the many people who helped make this thesis possible.

The author is indebted to his thesis advisor, Robert M. Dinnat, for allowing him the privilege of making mistakes and defending them vigorously; for out of mistakes comes knowledge, and out of student-faculty debate comes interest in learning.

The author wishes to thank the other members of his reading committee, Professor James R. Fincher and Dr. Paul H. Sanders, not only for their suggestions in the final phases of the preparation of the thesis, but also for their lectures and discussions which preceded this thesis.

The author is indebted to the Director of the School of Civil Engineering, Dr. F. W. Schutz, for his continued encouragement.

Financial assistance was provided by the National Science Foundation.

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SUMMARY

Second-order elastic equations, in series form, were derived to describe the elastic behavior of a segment of a beam column. These equations were combined with the equations of statics, compatibility and concentrated plasticity in order to predict the horizontal load and deflection that would cause a beam column or a simple frame, loaded with a known vertical load system, to collapse.

In order to avoid the somewhat laborious step-by-step method normally used in applying second-order elastic-plastic theory, a mechanism approach was employed. Each structure was analyzed at the instant when the final hinge of the mechanism was formed. This technique was adequate since only the load and deflection at collapse were desired.

A cantilever and several simple frames were studied by the above technique. The results were compared with other analytical and experimental studies. From such a comparison it was concluded that

1. Second-order elastic-plastic theory as utilized in this thesis is much easier to apply to beam-columns and simple frames than a rigorous compatibility analysis.

2. Results obtained from a second-order elastic-plastic analysis are slightly unconservative when compared with a compatibility analysis which takes into account residual stresses and the spread of plastification, but compare favorably with limited experimental work.

3. Further experimental work is needed to determine the exact relation between the behavior predicted by a second-order elastic-plastic solution and the true behavior.

4. The use of the series equations developed for second-order elastic theory gives a clear insight into the behavior of beam columns and frames subjected to combined vertical and horizontal loading.

SUMMARY OF NOMENCLATURE

A	-	cross sectional area of member
A_o	-	arbitrary constant in solution of differential equation
B_o	-	arbitrary constant in solution of differential equation
C	-	a constant
C_c	-	the sL/r that divides the buckling range, for $sL/r \geq C_c$ inelastic buckling controls, for $sL/r \leq C_c$ elastic buckling controls
E	-	modulus of elasticity
E_t	-	tangent modulus of elasticity
FY	-	yield stress
I	-	moment of inertia about the strong axis of section
I_1	-	moment of inertia of left vertical member of frame
I_2	-	moment of inertia of horizontal member of frame
I_3	-	moment of inertia of right horizontal member of frame
K	-	spring constant for column bases (kip-in/radian)
L	-	length of member
L_{ij}	-	length of beam column segment i-j
L_c	-	length of vertical member of frame
L_g	-	length of horizontal member of frame
M	-	moment
M_{ij}	-	moment at point i of beam-column segment i-j
M1	-	moment at base of left vertical member of frame
M2	-	moment at junction of left vertical member and horizontal member of frame

SUMMARY OF NOMENCLATURE (Continued)

M3 - moment at junction of right vertical member and horizontal member of frame

M4 - moment at base of right vertical member of frame

MH - moment at plastic hinge in horizontal member of frame

MPR - reduced moment capacity of member

MPR1 - reduced moment capacity at base of left vertical member in frame

MPR2 - reduced moment capacity at junction of left vertical and horizontal member in frame

MPR3 - reduced moment capacity at junction of right vertical and horizontal member in frame

MPR4 - reduced moment capacity at base of right vertical member in frame

MPRG - reduced moment capacity in horizontal member of frame

MP - moment capacity of member neglecting the effect of axial forces

M_y - moment at yield of outermost fiber of section

P - applied concentrated load

P_i - applied concentrated load number i

P_{el} - elastic buckling load

P_{in} - inelastic buckling thrust

Q - applied horizontal load

Q_o - applied horizontal load to cause mechanism as given by simple plastic theory

R - $\left[I_1/L_c\right] / \left[I_2/L_g\right]$, column-girder stiffness ratio

$S1_{(i,j)}$, $S2_{(i,j)}$, $S3_{(i,j)}$, $S4_{(i,j)}$, $S5_{(i,j)}$,

$S11_{(i,j)}$, $S12_{(i,j)}$, $S13_{(i,j)}$, $S14_{(i,j)}$,

$SG1_{(i,j)}$, $SG2_{(i,j)}$, $SG3_{(i,j)}$,

SUMMARY OF NOMENCLATURE (Continued)

$SG11_{(i,j)}$, $SG13_{(i,j)}$, $SG13_{(i,j)}$ - series expressions in terms of the parameter p for beam-column segment i-j

T - axial thrust

T_{ij} - axial thrust in beam-column segment i-j

$T1$ - axial thrust in left vertical member of frame

$T2$ - axial thrust in horizontal member of frame

$T3$ - axial thrust in right vertical member of frame

TY - axial thrust which causes entire cross section to yield

T' - $T \frac{T_{el}}{T_{in}}$, equivalent thrust used in modifying second-order elastic-plastic theory to include the effect of residual stresses

T_{el} - elastic buckling thrust

T_{in} - inelastic buckling thrust

V - shear force

V_{ij} - shear force at point i of beam-column segment i-j

$V1$ - shear force at base of left vertical member in frame

$V2$ - shear force at left end of horizontal member in frame

$V3$ - shear force at right end of horizontal member in frame

$V4$ - shear force at base of right vertical member in frame

VH - shear force to the right of a plastic hinge in the horizontal member of frame

a - a positive constant

b - flange width of a wide flange section

d - depth of a wide flange section

f - stress

SUMMARY OF NOMENCLATURE (Continued)

k - $\sqrt{T/EI}$

m - ratio of deflection calculated by second-order elastic theory to the deflection calculated by first-order elastic theory

p - $T_{ij}L_{ij}^2/E/I$, series parameter for beam column segment i-j

r - radius of gyration of section about its strong axis

s - effective length factor

t - flange thickness of wide flange section

w - web thickness for wide flange section

w_o - applied uniform load

x - distance measured along the neutral axis of member

y - deflection at any x

y' - first derivative of y with respect to x

y'' - second derivative of y with respect to x

∂_i - ratio of distance from left end of beam-column segment to total length of beam-column segment

ϵ - unit strain

ϵ_y - yield strain

Δ - deflection

Δ_{ij} - deflection of point j on beam-column segment relative to point i

Δo - deflection computed by first-order elastic theory

$\Delta 1$ - first estimate of deflection from iterative solution

$\Delta 2$ - second estimate of deflection from iterative solution

ϕ - curvature at section of beam-column segment

ϕ_y - curvature at section of beam column segment when outer fiber at the section yields

Θ - rotation

SUMMARY OF NOMENCLATURE (Continued)

θ_{ij} - rotation of point i on beam-column segment i-j

CHAPTER I

INTRODUCTION

The horizontal load-carrying capacity of a laterally unbraced steel building frame is reduced whenever it must simultaneously carry vertical loads. First-order elastic and plastic theory do not accurately account for the amount of this reduction. If one calls the vertical forces P and the amount of their horizontal deflection Δ, then secondary moments equal to $P \times \Delta$ occur causing deflections greater than those computed by first-order theory. These secondary moments and deflections are referred to as the $P - \Delta$ effect. The secondary moments may in certain cases be greater than the primary moments in the structure. This is particularly likely in the lower floors of multistory frames where the columns must carry heavy axial loads as well as large bending moments.

The horizontal load-carrying capacity of unbraced frames subjected to vertical loads has been the subject of both analytical (1)* and experimental (2) work in recent years. Methods of attacking the problem range from the highly theoretical work of Adams (3) to semi-empirical formulae such as Merchants' (4). The highly theoretical ones are so involved that for something as simple as a one-story frame the amount of computation time, even with a computer, is excessive. The somewhat simpler formulae are generally over-conservative, but there is no way to determine the true safety factor. A compromise between these two extremes is a

*Numbers in parentheses refer to references in "Literature Cited."

second-order elastic solution. This method very accurately takes into account the P - Δ effect, but it neglects the effect of residual stresses and strain hardening and concentrates all plasticity at discrete points called plastic hinges. Neglecting residual stresses and the spread of plastification is unconservative, but neglecting strain hardening is conservative; that the method gives good correlation to experimental work (2) is probably the result of their compensating effects.

Previous second-order elastic-plastic work done at Lehigh and elsewhere have their origin in the deflection equations derived by Timoshenko (5) for beams subjected simultaneously to lateral and axial loads. Timoshenko's equations are transcendental in terms of the loading and member stiffness. Because of their transcendental nature they are cumbersome to work with and do not give a clear insight into the behavior of individual beam-columns or frames. It was suggested by Robert M. Dinnat* that perhaps similar equations could be derived as series expansions of these parameters that would be more efficient for computation and also give a clearer insight into the behavior of structural components and frames. He derived many of the same equations derived herein by an independent approach.

The object of this thesis was the development of a computational technique that would predict, by second-order elastic-plastic theory, the magnitude of the horizontal load necessary to cause the collapse**

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*Thesis advisor.
**Collapse is defined herein as the formation of a mechanism.

of a simple frame, already subjected to a vertical load of known magnitude. It was hoped that the computational technique would be readily adaptable to programming on digital computers. It was felt that the ideas developed could be used in further work on larger and more complex structures.

CHAPTER II

VARIOUS THEORIES

Consider the simple frame shown below in Figure 1. It is subjected to a known vertical load system* of insufficient magnitude to cause it either to buckle or to fail by the formation of a beam mechanism, hence it will be able to withstand some horizontal load, Q, before collapsing.

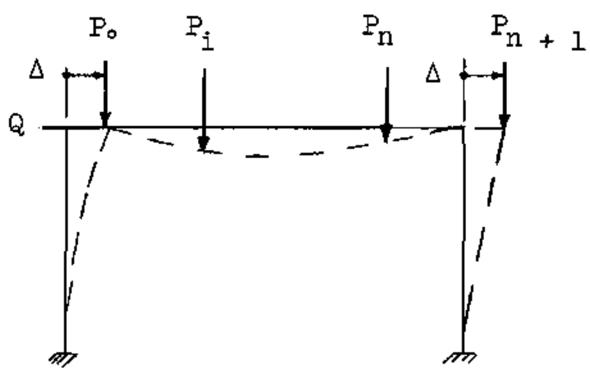


Figure 1. Rigid Frame in Deflected Position

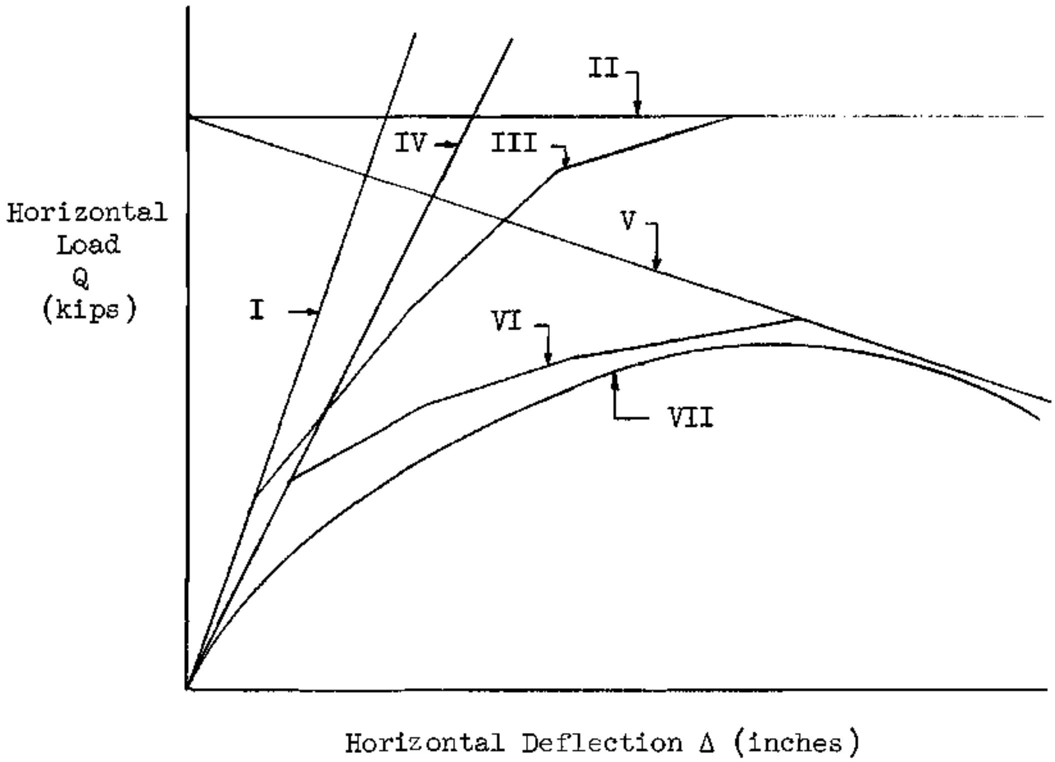
*The vertical load system is defined by the location and magnitude of the $n + 2$ vertical loads (P_i).

If the full vertical load were applied first and then the horizontal load were gradually increased, one would observe an increasing deflection, Δ . Analytical attempts to describe this load-deflection curve have been made by at least seven different theories, for which representative curves are shown in Figure 2. These theories are discussed separately in this chapter. In order to facilitate their discussion a brief review of the assumptions and limitations involved is included. It is assumed that the reader is familiar with simple elastic and plastic theory as applied to analysis and design of one-story, one-bay rigid frames.

General Assumptions

Common to all of the methods is the assumption that deflections are caused by the bending of the structure, hence deflections caused by shearing forces and axial deformation are ignored. Small deflection theory is assumed to be valid; thus for elastic portions of the structure the absolute value of the radius of curvature is given with sufficient accuracy by $1/y''$, rather than by the more exact formula from calculus (6). The symbol, y'', is the second derivative of deflection with respect to distance measured along the neutral axis of the member.

It is assumed that no out-of-plane deflections are permitted; hence, lateral-torsional (7) and weak-axis buckling are not permitted. In addition, compact sections (8) are required to prevent local buckling (7) of plate elements. With these assumptions failure is restricted to the plane of the frame and is caused by excessive bending.



Curve #	Theory
I	First-Order Elastic
II	First-Order Plastic
III	First-Order Elastic-Plastic
IV	Second-Order Elastic
V	Second-Order Plastic
VI	Second-Order Elastic-Plastic
VII	Compatibility

Figure 2. Horizontal Load-Deflection Curves for a Simple Bent by Seven Theories

Wide flange sections bent about the strong axis are the only ones considered in this thesis; however, modification of the basic ideas to other types of symmetrical sections would not involve any new concepts. All problems worked herein involve structural steel members; however, the extension of these ideas to any other material that possesses the stress-strain characteristics of steel should pose no analytical problems.

Independent Horizontal and Vertical Loading

The structures discussed in this thesis are loaded by two independent, proportional loading systems--one vertical and one horizontal. The horizontal load versus horizontal deflection curve (hereinafter referred to as the $Q - \Delta$ curve) is obtained by applying a fixed value of vertical load to the structure and then varying the horizontal load from zero to the failure load.*

Each value of vertical load will have a corresponding value of horizontal load that will cause failure. If one plots the corresponding values, at failure, of the vertical and horizontal loads, the resulting curve is called an interaction curve; it defines all possible pairs of vertical and horizontal loads that could cause failure - assuming, of course, that failure is independent of the loading path. Figure 27 shows interaction curves for a pinned base frame by second-order, elastic-plastic theory and by simple plastic theory.

*Failure load is defined herein for theories II, III, V, and VI to be the load required to form a mechanism. For Theories I and IV failure is usually defined with regard to stress whereas the failure load associated with theory VII is the maximum load on the load deflection curve.

<u>Stress-Strain Curve</u>

A linearly-elastic, perfectly-plastic stress-strain curve for any individual fiber is assumed, as shown in Figure 3, where f represents the stress, ϵ the unit strain, FY the yield stress, ϵ_y the yield strain, and E the Modulus of Elasticity. The curve is assumed to apply in tension and in compression. The pertinent values can be obtained experimentally either from a tension coupon test or a compression test of a very short annealed specimen. Simple plastic theory deviates from this stress-strain curve in that it assumes a state of zero strain until attainment of the yield stress after which perfect plastic flow takes place. All of the theories except number seven (Compatibility Analysis) consider the stress-strain curve for the entire section to be the same as for an individual fiber; hence, they neglect the effect of residual stresses and the spread of plastification. The beneficial effect of strain hardening has been ignored in almost all solutions. Reference No. 2 does consider the effect of strain hardening on post-mechanism behavior.

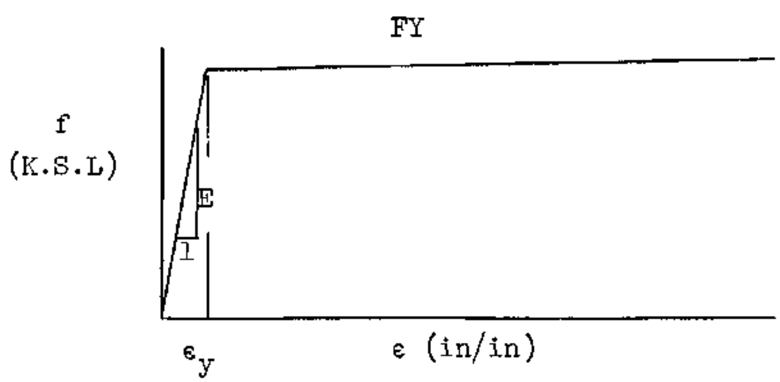


Figure 3. Assumed Stress-Strain Curve

First-Order Elastic Solution - Curve #1

Both stiffness and flexibility methods as commonly used on indeterminate structures are in this category. Statics is satisfied on the undeformed structure, so the P - Δ effect is ignored. The limit of usefulness of the structure is set when anywhere in the structure the stress, as calculated by the sum of the axial stress and the primary moment bending stress is equal to the yield stress; this postulates of course that no elastic buckling occurs prior to the limit.

In practice the stress level is further restricted to some allowable stress less than the yield stress so that the structure will have some safety factor. The method is conservative in that it ignores the ability of a frame to deform beyond initial yield and carry additional load, but it is unconservative in ignoring the P - Δ effect. Hence, the true safety factor is never known and could conceivably be less than one. Realizing this, codes attempt to account for the P - Δ effect in various ways such as magnification factors (8) which are used to increase the bending stresses obtained from the primary moments.

First-Order Plastic Solution - Curve #2

This is commonly referred to as simple plastic theory. The horizontal load that would cause collapse* is calculated so that three conditions are satisfied:

(1) Equilibrium;

(2) Plasticity; and

(3) Mechanism.

---

*The collapse load is considered to be synonymous with the mechanism load in this thesis.

Equilibrium

Equilibrium is satisfied on the undeformed structure. It may be satisfied by the equations of statics or by the equations of virtual work. When virtual work is used the structure is viewed as being on the verge of plastic deformation but the deflections cancel out of the equations. Hence, in either case, the P - Δ effect is ignored.

Plasticity

The reduced plastic moment capacity MPR is the plastic moment that a cross-section can carry in addition to an axial force. Its magnitude is "reduced" when compared to the plastic moment capacity MP of a cross-section carrying no axial force. The equations relating MPR to MP and the axial force are listed in Appendix A.

Mechanism

When sufficient hinges form to constitute a mechanism the structure is no longer able to withstand any increase in load and will deform freely under the load required to form the mechanism.

To calculate the true collapse load of the structure one must satisfy all three conditions simultaneously. This is generally done by assuming a mechanism, satisfying equilibrium and then checking for a violation of plasticity. Alternately, one may use an upper bound approach by calculating the collapse load for all possible mechanisms, the lowest of these being the collapse load.

Simple plastic theory takes into account the load-carrying capacity of the frame beyond initial yield but ignores the P - Δ effect; therefore, with large vertical loads it becomes very unconservative. For this reason its application is extremely restricted by building codes and manuals (8).

First-Order Elastic-Plastic Solution - Curve #3

This incorporates the ideas, and limitations, of the first two theories into a theory that allows the description of the entire load-deflection curve. All plasticity is concentrated at points of plastic hinges and the structure is assumed to behave elastically between these hinges.

An elastic analysis is performed on the structure and the load increased until the plastic moment capacity, MPR, is reached at one point. This point is the assumed to have MPR* acting and the structure is re-analyzed. The structure is now one degree less indeterminate. This process is continued until a mechanism is formed.

Mechanism Method

If one is interested only in the deflection at collapse, a mechanism technique is available (7) to find this deflection without resorting to a step-by-step procedure. Consider the behavior of the structure just prior to the formation of the last hinge. Hopefully it will be a stable, statically determinate structure. (This is the case for all structures investigated in this thesis.) Continuity of slope will be lacking at all of the hinges that have formed but will exist at the last hinge to form until the actual formation of the mechanism. If one knew which hinge formed last one could then calculate the deflection on a statically de-

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*MPR may change with the change in applied horizontal load due to the change in the axial forces in the members. A computer program for first-order elastic-plastic analysis of plane frames that takes this change in MPR into account has recently been developed at Lehigh (9).

terminate structure by assuming continuity at the last hinge to form and using any normal method of calculating deflections, such as virtual work. However, this means that all possible hinges in the mechanism must, in turn, be assumed to form last. The true deflection will be the greatest one so computed (7). This method, while time consuming, requires much less effort than a step-by-step analysis.

Second-Order Elastic Solution - Curve #4

A second-order elastic solution differs from a first-order elastic analysis in that statics is satisfied on the deformed structure. The moment variation is a function of the deflection which is not known. Therefore, the calculation of deflections is extremely complex.

Timoshenko (5) has solved the problem for several practical cases of beam columns. If, as mentioned in the introduction, Timoshenko's basic equations are expressed in series form rather than in transcendental form, one may write expressions for the deflection directly in terms of the forces in and on the structure and the E and I* of the members. If the structure is determinate, then one may immediately solve for the deflection. If the structure is indetrminate then it is necessary to set up a system of non-linear simultaneous equations of one** degree higher than the degree of static indeterminacy. These may be rapidly set up by this method but the solution of such a set of equations is a major task. The current approach to their solution is an iterative one. Chu (1) and others (2) have done this using the deflection equations of

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*I is the moment of inertia about the strong axis.

**The frames in this thesis have only one degree of deflection freedom.

Timoshenko, expressed in slope deflection form. Second-order elastic solutions for indeterminate structures were not attempted herein since it was found that the collapse load could be found without resorting to this.

For a beam-column under constant axial load the Q - ∇ curve will be linear but will be flatter than the Q - ∇ curve obtained from a first-order elastic analysis.

In frames where the axial forces in the members change as the horizontal load is increased, the Q - ∇ curve will be slightly non-linear.

Second-Order Plastic Solution - Curve #5

A second-order plastic solution differs from a first-order plastic solution in that equilibrium is formulated on the deforming structure. This gives the horizontal load capacity as a decreasing function of the deflection. The second-order plastic solution assumes a mechanism and hence is not valid until a mechanism forms. If one knew the deflection at the forming of the mechanism, the mechanism load could be obtained by solving the second-order plastic equation. One obtains a different second-order Q - ∇ equation for each possible mechanism.

One may obtain the second-order Q - ∇ equations by virtual work if the work done by the vertical forces is considered (10). They may also be obtained by applying statics to the deformed structure; this is the method used herein.

Second-Order Elastic-Plastic Solution - Curve #6

The combination of methods 4 and 5 in a step-by-step analysis similar to method 3 gives a sufficiently accurate prediction of the load deflection behavior of the frame. This opinion was given in lectures at Lehigh ([illegible]) and borne out by limited experimental (2) studies there. Further work on more complex frames is underway there at this date.

This method requires a second-order elastic analysis of the structure between the formation of successive hinges and has the difficulties inherent in second-order elastic solutions for indeterminate structures.

Mechanism Method

If only the deflection and load required to form a mechanism are desired the step-by-step procedure may be omitted as was done for a first-order elastic-plastic analysis. This greatly reduces the computation time involved. Lind (12) has worked problems by this technique, but uses Timoshenko's equations in a slightly modified form.

The equations derived herein in Chapter III work very well for such a mechanism method since they give deflection in terms of forces in and on the structure, all of which are either known or are, for a complete mechanism, statically determinable just prior to the formation of the last hinge. This allows one to solve directly for the deflection at collapse and then to use this deflection to calculate the mechanism load from the second-order plastic $Q - \Delta$ equation. For each possible mechanism the deflection must be obtained using the second-order elastic equations of Chapter III. As discussed under "First-Order Elastic-Plastic Solution" earlier in this chapter, continuity of slopes must be assumed at each possible last hinge point in turn, and the greatest deflection so computed will be the proper one to use in the $Q - \Delta$ relation for that particular mechanism. This procedure gives the Q and Δ required to form that particular mechanism. It must be remembered, however, in order to have the correct mechanism plasticity must still be satisfied.

Results obtained from the computer programs developed indicate that an upper bound solution, i.e., assuming all possible mechanisms and selecting the one that gives the lowest load, is not possible here as in a first-order elastic-plastic analysis. In some cases the energy required to move a small horizontal load through a large deflection in order to form one mechanism is greater than the energy required to move a larger horizontal load through a smaller deflection to form another mechanism. Since the mechanism requiring the least energy to form is the correct one, the larger load is the correct load. However, this poses no problem since a statics check on the smaller load would indicate a violation of plasticity whereas a statics check on the larger load would not.

In some cases the mechanism load may be slightly lower than the load when the next to the last hinge forms. This reduction is due primarily to the reduced moment capacity, MPR, in certain columns whose axial load increases as the horizontal load and deflection increase. In such cases the mechanism load is a conservative estimate of the maximum load.

Tangent-Modulus Modification of a Second-Order Elastic Plastic Solution. As the thesis progressed a semi-empirical modification to the second-order elastic-plastic theory was developed. It consists of using a Tangent Modulus* in the deflection equations rather than the normal Modulus of Elasticity. This has the effect of lowering the second-order elastic-plastic $Q - \Delta$ curve thereby moving closer to the $Q - \Delta$ curve

*The Tangent-Modulus was obtained from the average stress-strain curve for a short segment of a column that contains residual stresses.

obtained by a compatibility solution. The details of this Tangent-Modulus modification of a Second-Order Elastic-Plastic Solution are presented in Chapter IV.

Compatibility Solution - Curve #7

A more rigorous and time-comsuming method is available (1), (3), and differs from the second-order elastic-plastic analysis in that the effect of residual stresses and the spread of plastification are considered.

An exact analysis of statics and strain compatibility on a short segment of beam column indicates that the stress at a point is a complex function of the axial load, T*, the moment, M, the assumed residual stress distribution, the cross-section and the material properties. Hence, the only practical way to describe the curvature ϕ is by M - ϕ curves (13). M - ϕ curves for a typical section are shown in Figure 15. Also shown for comparison are the linear M - ϕ curves which result from the assumptions of second-order elastic-plastic theory. Because of the complicated nature of the non-linear compatibility curves, numerical techniques are necessary to describe the load deflection behavior for a structural element. Techniques for solutions of beam columns (1) and frames (1), (3) are available; however, the process of combining the M - ϕ equations with statics and with slope and deflection compatibility is so involved that even for the simplest problems large computer programs are necessary. Furthermore, because of the many iterative procedures involved, they are

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*Throughout this thesis, T refers to the Axial Thrust at a section, whereas P refers to an applied concentrated load. T may or may not be equal to P.

very time consuming even on high speed digital computers.

Since the only major assumption in this theory is the absence of strain hardening, which is conservative, it represents a true* lower bound solution for the load deflection curve. The real load deflection curve should lie between the compatibility solution and the second-order elastic-plastic solution.

Thesis Objective

After examining the details of each theory it was decided to pursue at length the second-order elastic-plastic solution, in particular, to utilize the mechanism method in determining the load-carrying capacity for a number of simple structures. While this method lacks the rigor of compatibility analysis, it was felt that results obtained from such an analysis could be compared with compatibility analysis and experimental results in the hope that it may prove to be an adequate** engineering analysis of a very complex problem.

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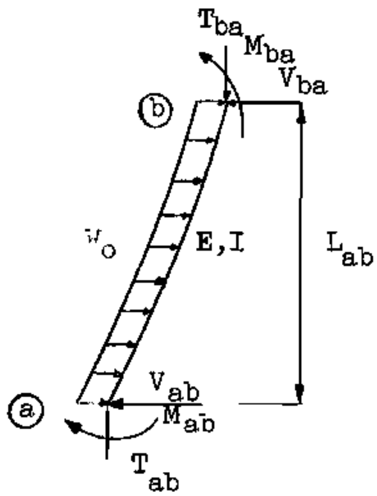
*Provided the general assumptions outlined at the beginning of the chapter are valid.

**Adequate in that it may be performed in a reasonable amount of time, and give answers that are close to the true behavior.

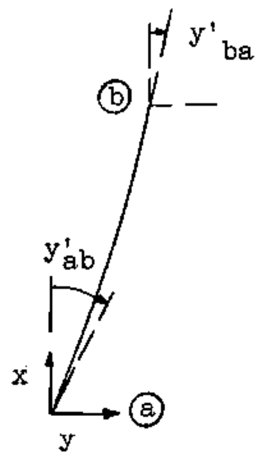
CHAPTER III

DEVELOPMENT OF THE SECOND-ORDER ELASTIC DEFLECTION EQUATIONS

A segment of a beam column, prismatic between points a and b, was investigated as shown in Figure 4. The segment was investigated in its final deflected shape, in equilibrium under the action of its applied loads and reactions.



4a
Force Systems



4b
Coordinates

Figure 4. Forces and Coordinates for a Segment of a Beam-Column

The segment has a uniform load w_o applied over its entire length L_{ab}. It has a constant moment of inertia, I, and a constant modulus of elasticity, E. A shear force, V_{ab}, a moment M_{ab}, and an axial thrust T_{ab}, which is constant from a to b, are required at a to keep the segment in equilibrium. All forces and moments are positive as shown in Figure 4a. This is the same sign convention as normally used in engineering

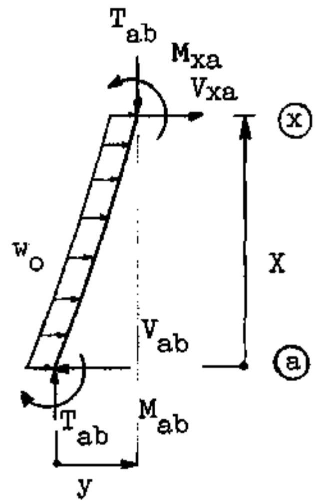


Figure 5. Freebody of a Beam-Column Segment

mechanics.* The segment has coordinates positive as shown in Figure 4b, and slopes y'_{ab} and y'_{ba} at points a and b respectively. These slopes are also positive as shown; this is in accordance with the normal calculus sign convention. For a sign convention of a complete frame see Figure 16.

The segment was assumed to be linearly elastic, and no concentrated loads were assumed to act between points a and b. However, plasticity and concentrated loads would not be disallowed at the ends of the segment.

For such an elastic segment it is possible to define the deflection, y and the slope, y', at any point, x, between a and b, in terms of w_o, E, I, V_{ab}, T_{ab}, M_{ab}, L_{ab}, y'_{ab}, and the distance x from a to point x.

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*The engineering mechanics sign convention assigns positive value to a moment which causes tension on one side of a beam rather than by which direction it rotates either on, the member or on a joint, as is done in slope deflection or moment distribution.

To find this relationship it was necessary to investigate the governing differential equation of bending:

$$y'' = -\frac{M_{xa}}{E\ I} \tag{1}$$

M_{xa} is the moment at any point x as shown in the freebody in Figure 5 and is given by:

$$M_{xa} = M_{ab} + V_{ab}\ x + T_{ab}\ y - \frac{w_o x^2}{2} \tag{2}$$

Letting

$$k^2 = \frac{T_{ab}}{E\ I} \tag{3}$$

and substituting (2) and (3) into (1) gives

$$y'' + k^2 y = -\frac{M_{ab}k^2}{T_{ab}} - \frac{V_{ab}k^2 x}{T_{ab}} + \frac{w_o k^2 x^2}{2T_{ab}} \tag{4}$$

The well known complementary solution to equation (4) is

$$y = A_o \sin\ (kx) + B_o \cos\ (kx) \tag{5}$$

A_o and B_o are arbitrary constants to be determined by the boundary conditions. The particular solution is

$$y = -\frac{M_{ab}}{T_{ab}} - \frac{V_{ab}x}{T_{ab}} + \frac{w_o x^2}{2T_{ab}} - \frac{w_o}{T_{ab}k^2} \tag{6}$$

The complete solution is the sum of (5) and (6).

$$y = A_o \sin\ (kx) + B_o \cos\ (kx) - \frac{M_{ab}}{T_{ab}} - \frac{V_{ab}x}{T_{ab}} + \frac{w_o x^2}{2T_{ab}} - \frac{w_o}{T_{ab}k^2} \tag{7}$$

To solve for A_o and B_o one may use the following independent boundary conditions:

when $\qquad x = 0, \quad y = 0 \qquad (8)$

and when $x = 0, \quad y' = y'_{ab}$ (9)

Using these boundary conditions to solve for A_o and B_o and then substituting these expressions into (7) gives

$$y = (y'_{ab} + \frac{V_{ab}}{T_{ab}}) \frac{\sin (kx)}{k} + (\frac{M_{ab}}{T_{ab}} + \frac{w_o}{T_{ab}k^2}) \cos (kx) - \frac{M_{ab}}{T_{ab}} - \frac{V_{ab}x}{T_{ab}} + \frac{w_o x^2}{2T_{ab}} - \frac{w_o}{T_{ab}k^2} \tag{10}$$

Expanding sin (kx) and cos (kx) in their well known series forms and canceling appropriate terms gives

$$\begin{aligned} y = {} & y'_{ab} (x - \frac{k^2x^3}{3!} + \frac{k^4x^5}{5!} - \frac{k^6x^7}{7!} + \ldots) \\ & - (\frac{M_{ab}}{T_{ab}}) (\frac{k^2x^2}{2!} - \frac{k^4x^4}{4!} + \frac{k^6x^6}{6!} - \frac{k^8x^8}{8!} + \ldots) \\ & - (\frac{V_{ab}}{T_{ab}}) (\frac{k^2x^3}{3!} - \frac{k^4x^5}{5!} + \frac{k^6x^7}{7!} - \frac{k^8x^9}{9!} + \ldots) \\ & + (\frac{w_o}{T_{ab}}) (\frac{k^2x^4}{4!} - \frac{k^4x^6}{6!} + \frac{k^6x^8}{8!} - \frac{k^8x^{10}}{10!} + \ldots) \end{aligned} \tag{11}$$

Differentiating equation (11) term by term with respect to x gives

$$\begin{aligned} y' = {} & y'_{ab} (1 - \frac{k^2x^2}{2!} + \frac{k^4x^4}{4!} - \frac{k^6x^6}{6!} + \ldots) \\ & - (\frac{M_{ab}}{T_{ab}}) (k^2x - \frac{k^4x^3}{3!} + \frac{k^6x^5}{5!} - \frac{k^8x^7}{7!} + \ldots) \end{aligned} \tag{12}$$

$$- \left(\frac{V_{ab}}{T_{ab}}\right) \left(\frac{k^2x^2}{2!} - \frac{k^4x^4}{4!} + \frac{k^6x^6}{6!} - \frac{k^8x^8}{8!} + \ . \ . \ .\right)$$

$$+ \left(\frac{w_o}{T_{ab}}\right) \left(\frac{k^2x^3}{3!} - \frac{k^4x^5}{5!} + \frac{k^6x^7}{7!} - \frac{k^8x^9}{9!} + \ . \ . \ .\right) \qquad (12a)$$

Letting

$$p = \frac{T_{ab}x^2}{E\ I} \qquad (13)$$

and substituting (13) and (3) into (11) and (12) gives

$$y = y'_{ab}x \left(1 - \frac{p}{3!} + \frac{p^2}{5!} - \frac{p^3}{7!} + \ . \ . \ .\right)$$

$$- \left(\frac{M_{ab}x^2}{E\ I}\right) \left(\frac{1}{2} - \frac{p}{4!} + \frac{p^2}{6!} - \frac{p^3}{8!} + \ . \ . \ .\right)$$

$$- \left(\frac{V_{ab}x^3}{EI}\right) \left(\frac{1}{3!} - \frac{p}{5!} + \frac{p^2}{7!} - \frac{p^3}{9!} + \ . \ . \ .\right) \qquad (14)$$

$$+ \left(\frac{w_o x^4}{EI}\right) \left(\frac{1}{4!} - \frac{p}{6!} + \frac{p^2}{8!} - \frac{p^3}{10!} + \ . \ . \ .\right)$$

and

$$y' = y'_{ab} \left(1 - \frac{p}{2} + \frac{p^2}{4!} - \frac{p^3}{6!} + \ . \ . \ .\right)$$

$$- \left(\frac{M_{ab}x}{E\ I}\right) \left(1 - \frac{p}{3!} + \frac{p^2}{5!} - \frac{p^3}{7!} + \ . \ . \ .\right)$$

$$- \left(\frac{V_{ab}x^2}{E\ I}\right) \left(\frac{1}{2} - \frac{p}{4!} + \frac{p^2}{6!} - \frac{p^3}{8!} + \ . \ . \ .\right) \qquad (15)$$

$$+ \left(\frac{w_o x^3}{E\ I}\right) \left(\frac{1}{3!} - \frac{p}{5!} + \frac{p^2}{7!} - \frac{p^3}{9!} + \ . \ . \ .\right)$$

If $p = 0$ in the series of equations (14) and (15), which corresponds to zero axial load, the only terms that remain are the first terms of each series. These are the identical terms that would be obtained if the analysis had been carried out on a beam with no axial load.

These series also hold for a beam column in tension rather than in compression if we assign a tension load a negative sign. This will then make p negative and make all terms of the series positive. A derivation similar to the one preceding this was made for a tension force T_{ab} instead of a compressive force T_{ab}. The solution of the complementary part of the differential equation came out in sinh(kx) and cosh (kx) rather than in sin (kx) and cos (kx). However, when the results were expanded in series form exactly the same series as in equations (14) and (15) resulted except that all signs were positive. Hence, if a tension force is assigned a negative value, equations (14) and (15) automatically take care of this change in the series.

Examining the series by the ratio test indicates that they are convergent for all values of p. Any divergence that may occur in working with these series will result from operations* with these series rather than from the series themselves.

The non-dimensional parameter p may be shown to be related to the elastic buckling thrust T_{el}**, as follows. T_{el} is given by the well known buckling formula as

---

*One such operation would be dividing by a series that is equal to zero.

**T_{el} refers to the value of the axial force which would cause a member to buckle. P_{el} is used to refer to the applied load that would cause a member to buckle.

$$T_{el} = \frac{\pi^2 E I}{(sL_{ab})^2} \tag{16}$$

where s is the effective length factor of the member. Substituting this expression into (13) gives

$$p = \left(\frac{\pi}{s}\right)^2 \left(\frac{x}{L_{ab}}\right)^2 \left(\frac{T_{ab}}{T_{el}}\right) \tag{17}$$

The series parameter p is also a function of the ratio T/TY. TY is the yield load for the member and is equal to the product of yield stress FY and the cross sectional area A. Since

$$I = r^2 A \tag{18}$$

and

$$TY = FY(A) \tag{19}$$

p is given by equation (13) as

$$p = \left(p\frac{X}{r}\right)^2 \left(\frac{FY}{E}\right) \left(\frac{T_{ab}}{TY}\right) \tag{20}$$

Series Notation

To facilitate writing equations with the series, a series notation was developed. Calling the deflection of point b relative to point a, Δ_{ba}, and the rotations y'_{ab} and y'_{ba}, θ_{ab} and θ_{ba}, equations (14) and (15) can be rewritten as

$$\Delta_{ba} = \theta_{ab} L_{ab} S2_{(a,b)} - \frac{M_{ab} L_{ab}^2}{E I} S3_{(a,b)} \tag{21}$$

$$- \frac{V_{ab} L_{ab}^3}{EI} S4_{(a,b)} + \frac{w_o L_{ab}^4}{EI} S5_{(a,b)}$$

and

$$\theta_{ba} = \theta_{ab} S1_{(a,b)} - \frac{M_{ab}L_{ab}}{EI} S2_{(a,b)} - \frac{V_{ab}L_{ab}^{2}}{EI} S3_{(a,b)} + \frac{w_o L_{ab}^{4}}{EI} S4_{(a,b)} \tag{22}$$

where

$$S1_{(a,b)} = 1 - \frac{p}{2} + \frac{p^2}{4!} - \frac{p^3}{6!} + \ldots = \sum_{i=0}^{\infty} \frac{(-p)^i}{2i!} \tag{23}$$

$$S2_{(a,b)} = 1 - \frac{p}{3!} + \frac{p^2}{5!} - \frac{p^3}{7!} + \ldots = \sum_{i=0}^{\infty} \frac{(-p)^i}{(2i+1)!} \tag{24}$$

$$S3_{(a,b)} = \frac{1}{2} - \frac{p}{4!} + \frac{p^2}{6!} - \frac{p^3}{8!} + \ldots = \sum_{i=0}^{\infty} \frac{(-p)^i}{(2i+2)!} \tag{25}$$

$$S4_{(a,b)} = \frac{1}{3!} - \frac{p}{5!} + \frac{p^2}{7!} - \frac{p^3}{9!} + \ldots = \sum_{i=0}^{\infty} \frac{(-p)^i}{(2i+3)!} \tag{26}$$

$$S5_{(a,b)} = \frac{1}{4!} - \frac{p}{6!} + \frac{p^2}{8!} - \frac{p^3}{10!} + \ldots = \sum_{i=0}^{\infty} \frac{(-p)^i}{(2i+4)!} \tag{27}$$

A computer program was developed that calculated the values of each of these series using enough terms that the change brought about by adding another term caused an insignificant change in the sum of all the terms. This program was used to print out values of the five basic series versus the parameter, p. These tables were then used in hand computations. A small table of these series is given in Appendix E. In addition, the program was made into a procedure that was used in subsequent programs. The series parameter, p, between points a and b on a beam column is given by

$$p = \frac{T_{ab} L_{ab}^{2}}{EI} \tag{28}$$

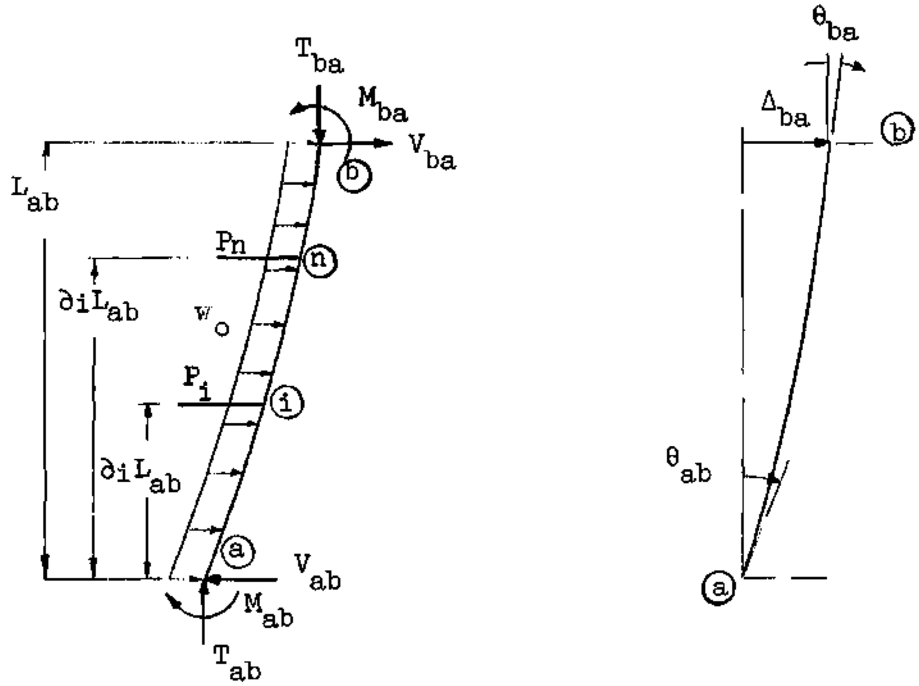
where L_{ab} is the distance between points a and b. Points a and b can be any two points on a beam column which have no concentrated loads, no plasticity, and no change in E or I between these two points. If E or I varies with the length, an incremental analysis can be performed using short segments wherein the E and I are assumed to be constant.

Concentrated Loads

Timoshenko (5) has proposed a limited form of superposition that holds for beam columns. If the axial force in a beam column is kept constant the deflection for any bending effect is proportional to that effect. This is apparent from examining equations (21) and (22). Each primary bending term such as $-(M_{ab} L_{ab}^{2}/(EI)$ is multiplied by a series multiplier which is a function of the axial force. If the axial force is kept constant, then the deflections will be proportional to the primary bending effects.

Making use of this limited form of superposition one can find the deflection if a series of concentrated loads is placed between points a and b as is shown in Figure 6. First, consider only one concentrated load acting, P_i.

To calculate the deflection at some point, say b, on the other side of the concentrated load, one considers that the effects of w_o, M_{ab}, V_{ab}, T_{ab}, and θ_{ab}, as given by equation (21), have not changed by the existence of the concentrated load although some of the magnitudes of these forces or of the rotation may have been changed. However, there is now the additional force, P_i, which causes some additional



6a

Force Systems

6b

Slopes and Deflections

Figure 6. Force Systems, Slopes and Deflections for a Segment of a Beam-Column with Concentrated Lateral Loads Between its Ends

deflection. Noting that it is the same as V_{ab} but that it acts in the opposite direction and on a shorter length $(1 - \partial_i)\, L_{ab}$ between i and b, the deflection at b relative to a is given by

$$\Delta_{ba} = \text{Equation (21)} + \left(\frac{P_i\, (1 - \partial_i)^3\, L_{ab}^{\;3}}{EI}\right) S4_{(i,b)} \qquad (29)$$

In fact, if n concentrated loads act between points a and b the deflection and rotation at b are given by

$$\Delta_{ba} = \theta_{ab} L_{ab} S2_{(a,b)} - \left(\frac{M_{ab} L_{ab}^{2}}{EI}\right) S3_{(a,b)} - \left(\frac{V_{ab} L_{ab}^{3}}{EI}\right) S4_{(a,b)} + \left(\frac{w_o L_{ab}^{4}}{EI}\right) S5_{(a,b)} + \sum_{i=1}^{n} \left(\frac{P_i (1 - \partial_i)^3 L_{ab}^{3}}{EI}\right) S4_{(i,b)} \tag{30}$$

and

$$\theta_{ba} = \theta_{ab} S1_{(a,b)} - \left(\frac{M_{ab} L_{ab}}{EI}\right) S2_{(a,b)} - \left(\frac{V_{ab} L_{ab}^{2}}{EI}\right) S3_{(a,b)} + \left(\frac{w_o L_{ab}^{3}}{EI}\right) S4_{(a,b)} + \sum_{i=1}^{n} \left(\frac{P_i (1 - \partial_i)^2 L_{ab}^{2}}{EI}\right) S3_{(i,b)} \tag{31}$$

The above line of reasoning is apparent with hindsight but was not so apparent in the development stages of this thesis. Therefore, the deflection at b was found by using equation (21) between points a and i and then reapplying it between i and b, adding Δ_{ib} to Δ_{ai} to get Δ_{ab}. This procedure yielded an equation identical to (30), thus confirming the principle of superposition for beam columns under constant axial load.

These two equations suffice to define the deflection response of an elastic portion of a beam column. However, sometimes it is advantageous to have the deflection at b in terms of the rotation at b*

---

*The slope at b may be known to be equal to the slope of a girder which is rigidly connected to the beam column at b.

rather than the rotation at a. This relationship may be obtained by solving equation (31) for θ_{ab} and then substituting this into equation (30). This gives

$$\begin{aligned}\Delta_{ba} = {} & \theta_{ba} L_{ab} \frac{S2_{(a,b)}}{S1_{(a,b)}} + \frac{M_{ab} L_{ab}^2}{EI} \left(\frac{S2_{(a,b)}}{S1_{(a,b)}} S2_{(a,b)} - S3_{(a,b)}\right) \\ & + \frac{V_{ab} L_{ab}^3}{EI} \left(\frac{S2_{(a,b)}}{S1_{(a,b)}} S3_{(a,b)} - S4_{(a,b)}\right) \\ & - \frac{w_o L_{ab}^4}{EI} \left(\frac{S2_{(a,b)}}{S1_{(a,b)}} S4_{(a,b)} - S5_{(a,b)}\right) \\ & - \sum_{i=1}^{n} \frac{P_i (1 - \partial_i)^3 L_{ab}^3}{EI} \left(\frac{S2_{(a,b)}}{S1_{(a,b)}} S3_{(i,b)} - S4_{(i,b)}\right)\end{aligned} \tag{32}$$

Letting

$$S11_{(a,b)} = \frac{S2_{(a,b)}}{S1_{(a,b)}} \tag{33}$$

$$S12_{(a,b)} = \frac{S2_{(a,b)}}{S1_{(a,b)}} S2_{(a,b)} - S3_{(a,b)} \tag{34}$$

$$S13_{(a,b)} = \frac{S2_{(a,b)}}{S1_{(a,b)}} S3_{(a,b)} - S4_{(a,b)} \tag{35}$$

and

$$S14_{(a,b)} = \frac{S2_{(a,b)}}{S1_{(a,b)}} S4_{(a,b)} - S5_{(a,b)} \tag{36}$$

gives

$$\Delta_{ba} = \theta_{ba} L_{ab}\, S11_{(a,b)} + \frac{M_{ab} L_{ab}^{2}}{EI} S12_{(a,b)}$$

$$+ \frac{V_{ab} L_{ab}^{3}}{EI} S13_{(a,b)} - \frac{w_o L_{ab}^{4}}{EI} S14_{(a,b)} \qquad (37)$$

$$- \sum_{i=1}^{n} \frac{P_i (1 - \partial_i)^3}{EI} \left(S11_{(a,b)} S3_{(i,b)} - S4_{(i,b)}\right)$$

Expressions for $S11_{(a,b)}$, $S12_{(a,b)}$, $S13_{(a,b)}$, and $S14_{(a,b)}$ can be obtained by multiplication, division and subtraction of the basic series. The expressions so obtained are listed in Appendix C. While such expressions are valuable for studying the effects of the parameter p, it will generally be easier to calculate values for the five basic series separately and combine them algebraically.

Girder Slopes

Most investigators have neglected the effect of the axial force on the stiffness of the girder. Equations are derived herein which include that this effect is small when the parameter p for the girder is small.

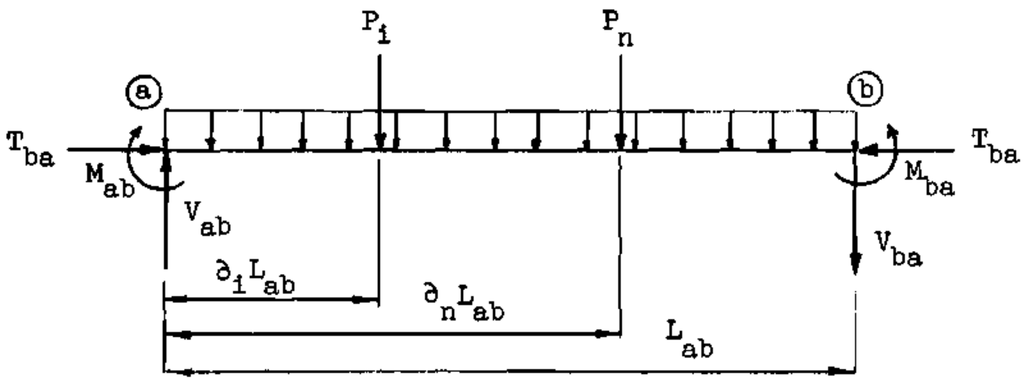
Consider the girder shown in Figure 7, both of whose ends are at the same elevation and whose material is elastic throughout.* For such a girder one may write equation (30) for Δ_{ba}, but Δ_{ba} is equal to zero since both ends have the same elevation. Solving this expression for θ_{ab} gives

*The possibility of plastic hinges forming in a girder is considered in Chapter VI.

$$\theta_{ab} = \left(\frac{M_{ab}L_{ab}}{EI}\right)\frac{S3_{(a,b)}}{S2_{(a,b)}} + \left(\frac{V_{ab}L_{ab}^{\;2}}{EI}\right)\frac{S4_{(a,b)}}{S2_{(a,b)}}$$

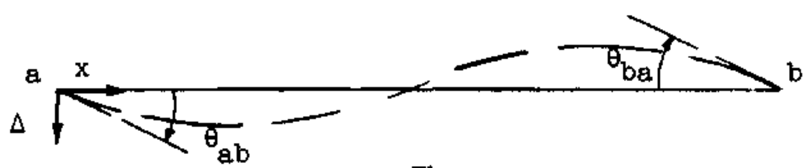
$$- \left(\frac{w_{\partial}ab^3}{EI}\right)\frac{S5_{(a,b)}}{S2_{(a,b)}}$$

$$- \sum_{i=1}^{n}\left(\frac{P_i\,(1-\partial_i)^3\,L_{ab}^{\;2}}{EI}\right)\frac{S4_{(i,b)}}{S2_{(a,b)}}$$



7a

Force Systems



7b

Slopes and Deflections

Figure 7. Force Systems, Slopes, and Deflections for a Girder

Letting

$$SG1^*_{(a,b)} = \frac{S3_{(a,b)}}{S2_{(a,b)}} \tag{39}$$

$$SG2^*_{(a,b)} = \frac{S4_{(a,b)}}{S2_{(a,b)}} \tag{40}$$

and

$$SG3^*_{(a,b)} = \frac{S5_{(a,b)}}{S2_{(a,b)}} \tag{41}$$

gives

$$\begin{aligned} \theta_{ab} &= \frac{M_{ab}L_{ab}\ SG1_{(a,b)}}{EI} + \frac{V_{ab}L_{ab}^{\ 2}}{EI} SG2_{(a,b)} \\ &- \frac{w_oL_{ab}^3}{EI} SG3_{(a,b)} - \sum_{i=1}^{n} \frac{P_i(1 - \partial_i)^3\ L_{ab}^{\ 2}}{EI} \frac{S4_{(i,b)}}{S2_{(a,b)}} \end{aligned} \tag{42}$$

θ_{ba} can be obtained by combining equations (30) and (31) to give

$$\begin{aligned} \theta_{ba} = &- \frac{M_{ab}L_{ab}}{EI} \left(S2_{(a,b)} - \frac{S3_{(a,b)}}{S2_{(a,b)}} S1_{(a,b)}\right) \\ &- \frac{V_{ab}L_{ab}^{\ 2}}{EI} \left(S3_{(a,b)} - \frac{S4_{(a,b)}}{S2_{(a,b)}} S1_{(a,b)}\right) \\ &+ \frac{w_oL_{ab}^3}{EI} \left(S4_{(a,b)} - \frac{S5_{(a,b)}}{S2_{(a,b)}} S1_{(a,b)}\right) \end{aligned} \tag{43}$$

---

*The first three terms of these series are listed in Appendix C.

$$+ \sum_{i=1}^{n} \frac{P_i\ (1 - \partial_i)^2\ L_{ab}^{\ 2}}{EI} \left(S3_{(i,b)} - (1 - \partial_i) \frac{S4_{(i,b)}\ S1_{(a,b)}}{S2_{(a,b)}} \right) \tag{43a}$$

Letting

$$SG11^*_{(a,b)} = S2_{(a,b)} - \frac{S3_{(a,b)}}{S2_{(a,b)}} S1_{(a,b)} \tag{44}$$

$$SG12^*_{(a,b)} = S3_{(a,b)} - \frac{S4_{(a,b)}}{S2_{(a,b)}} S1_{(a,b)} \tag{45}$$

$$SG13^*_{(a,b)} = S4_{(a,b)} - \frac{S5_{(a,b)}}{S2_{(a,b)}} S1_{(a,b)} \tag{46}$$

Then (43) becomes

$$\begin{aligned} \theta_{ba} = &- \frac{M_{ab} L_{ab}}{EI} SG11_{(a,b)} - \frac{V_{ab} L_{ab}^{\ 2}}{EI} SG12_{(a,b)} \\ &+ \frac{w_o L_{ab}^{\ 3}}{EI} SG13_{(a,b)} \\ &+ \sum_{i=1}^{n} \frac{P_i\ (1 - \partial_i)^2\ L_{ab}^{\ 2}}{EI} \left(S3_{(i,b)} - (1 - \partial_i) \frac{S4_{(i,b)}\ S1_{(a,b)}}{S2_{(a,b)}} \right) \end{aligned} \tag{47}$$

---

*The first three terms of these series are listed in Appendix C.

Important Equations

Equations (23) through (28), (30), (31), (33) through (37), (39) through (42) and (44) through (47) are used for obtaining the elastic response of elastic and elastic-plastic structures.

CHAPTER IV

CANTILEVER BEAM COLUMN

The cantilever beam column is the simplest structure that can be used to study the P - Δ effect; yet, the analysis of this simple problem greatly increases the understanding of second-order elastic-plastic behavior. In addition, it provides an exact mathematical solution against which numerical techniques can be checked.

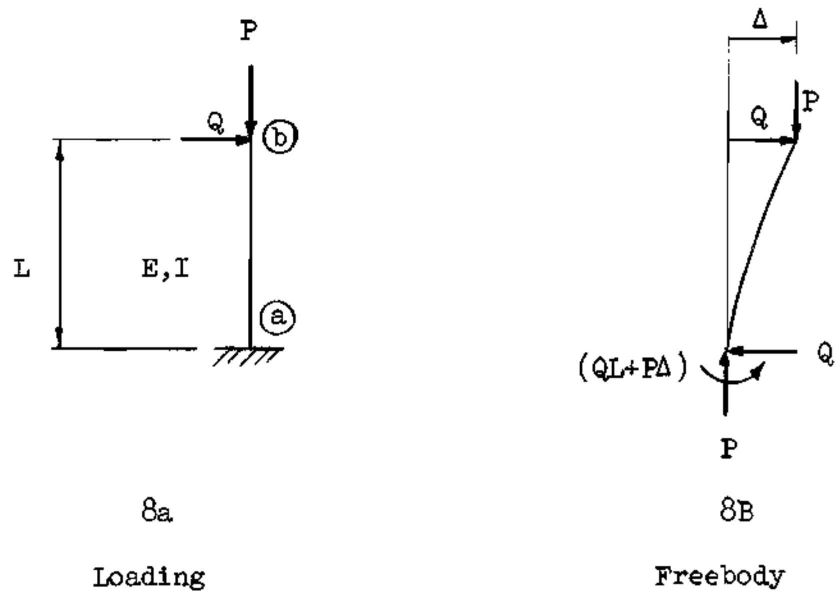


Figure 8. Loading and Freebody of a Cantilever Beam-Column

Second-Order Elastic Behavior

The elastic behavior of the cantilever beam column shown in Figure 8 was investigated using the appropriate series equations developed in Chapter III.

Second-Order Elastic Q - Δ Curve

Consider the elastic behavior of the cantilever beam column under the action of some P load less than P_{e1}. It is also necessary that the vertical load be less than the yield load for the column so that some elastic behavior will exist.

The load P is first applied and then Q is applied in increments. It is noted that the horizontal load deflection curve Q - Δ is linear but that it has a smaller slope than it would have if the cantilever had no axial load on it. Figure 17 shows the elastic load deflection curve for an 8WF31 column for various values of the parameter, p. How these curves were obtained is explained in the following.

Figure 8b shows the reactive elements required at the base of the cantilever. Using these and recognizing that θ_{ab} equals zero, equation (18) gives:

$$\Delta = \frac{(QL + P\Delta)\ L^2}{EI}\, S3^{*} - \frac{QL^3}{EI}\, S4 \tag{48}$$

The axial thrust at the base of the cantilever T_{ab} is equal to the applied vertical load P hence the series parameter p is given by

$$p = \frac{PL^2}{EI} \tag{49}$$

---

*The involved series notation outlined in Chapter III is not necessary for this simple case and is omitted in this chapter.

Rearranging and substituting (49) into (48) gives

$$\Delta = \frac{QL^3}{EI} (S3 - S4) + p\Delta\, S3 \qquad (50)$$

Solving for Δ gives

$$\Delta = \frac{QL^3}{EI} \frac{(S3 - S4)}{1 - pS3} \qquad (51)$$

Expanding S3 and S4 in terms of p and performing the necessary operations* gives

$$\Delta = \frac{QL^3}{EI} \left(\frac{1}{3} + \frac{2p}{15} + \frac{17p^2}{315} + \frac{62p^3}{2835} + \ldots\right) \qquad (52)$$

Or

$$\Delta = \frac{QL^3}{3EI} \left(1 + \frac{2p}{5} + \frac{17p^2}{105} + \frac{62p^3}{945} + \ldots\right) \qquad (53)$$

This gives the deflection as Δ_o (the deflection given by a first-order elastic analysis) times m, the deflection magnifier, which is also given from equation (51) as

$$m = \frac{3(S3 - S4)}{1 - pS3} \qquad (54)$$

Equation (53) indicates that it is indeed a linear Q - Δ relation for any given value of axial load, and the higher the axial load the greater the deflection for a given Q.

---

*See Appendix C for these series operations which are representative of the series operations performed to obtain many expressions in this thesis.

**It will be shown later that p = 2.47 corresponds to the elastic buckling load of the column. Therefore p = 1 represents an axial load of approximately forty percent of the buckling load.

When p is equal to 1 the first four terms in the series of equation (53) give an m of 1.63. Using equation (54) and the tabulated values of series S3 and S4 gives an m of 1.67. Therefore using the first four terms of the series gives an m within approximately 2-1/2 per cent of the true value. In general as p increases, more terms are required to calculate m accurately. In fact, as P approaches P_{el} the series will diverge. (For the cantilever P_{el} and T_{el} are identical.)

Iterative Solution. Although it was easy to get the closed form solution for this case, an iterative solution could be performed as outlined below.

1. Calculate all forces* and moments assuming Δ equals zero.

2. Calculate deflections using calculated forces.

3. Recalculate forces and moments that are functions of Δ with the new Δ.

4. Repeat steps 2 and 3 until a cycle produces an insignificant change in Δ.

How this method works can be seen by examining equation (50). Rewriting it in terms of p gives

$$\Delta = \frac{QL^3}{EI}\left(\frac{1}{3} - \frac{p}{30} + \frac{p^2}{840} - \frac{p^3}{45,360}\right) + P\Delta\left(\frac{1}{2} - \frac{p}{24} + \frac{p^2}{720}\right) \qquad (55)$$

---

*In a frame all or most of the forces and moments will be functions of the deflection Δ. If one neglects this deflection in calculating the shear and moment and does not iterate as in steps 2 and 3, the solution for Δ will be off in the second term of its series solution. Neglecting the change due to Δ in an axial force has one order of magnitude less effect on the deflection calculation. That is, it does not affect the second term of the series solution for Δ but it does affect the third term.

The iterative approach just outlined is equivalent to solving equation (55) by successive approximations.

1. Set $\Delta = 0$ on the right side of the equation.

2. Solve for Δ.

3. Using the Δ of step 2 in the right side of the equation recalculate Δ.

4. Repeat steps 2 and 3 until convergence occurs.

In our case, letting $\Delta = 0$ gives

$$\Delta = \frac{QL^3}{3EI}\,(1 - \ldots) \quad \text{(or the first term of this series is the same as the first term of equation (53))} \tag{56}$$

Then substituting this value for Δ on the right side of equation (55) gives

$$\Delta = \frac{QL^3}{EI}\left(\frac{1}{3} - \frac{p}{30} + \ldots\right) + \frac{QL^3}{3EI}\,p\left(\frac{1}{2} - \ldots\right) \tag{57}$$

or,

$$\Delta = \frac{QL^3}{3EI}\left(1 + \frac{2p}{5} - \ldots\right) \quad \text{(or the first two terms of this series are the same as the first two terms of the series of equation (53))} \tag{58}$$

Continuing this operation shows that the series can be built up term by term and that the iterative technique works as long as the series one is working with is convergent,* in other words, if P is less than P_{el}. Any numerical error in this recursion process will be in terms of higher order than the last term obtained.

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*One special case arose in working with frames whereby this iterative procedure failed to converge even though the series converged; this is discussed in Chapter V.

Elastic Buckling by the Ratio Test

The iterative solution works unless each additional term of the magnification series has a magnitude equal to or greater than the preceding one. In other words, divergence occurs if the ratio of the $n + 1$th term to the nth term is greater than or equal to one. The elastic buckling load could be found then by setting the ratio of the $n + 1$th term to the nth term equal to one; however, obtaining an expression for the nth term for even this case is not simple. An estimate of the buckling load can be made by setting the ratio of any two adjacent terms equal to one. If the ratio of the second term to the first term is set equal to one, p is found to be 2.50. Setting the ratio of the third term to the second term equal to one gives 2.47. Continuing this procedure for all remaining pairs of terms gives 2.47. Actually, if enough significant figures were carried it would be found that the answer was approaching $\pi^2/4$, which is 2.47 to three significant figures. If

$$p = \frac{\pi^2}{4} \tag{59}$$

then

$$\frac{\pi^2}{4} = \frac{PL^2}{EI} \tag{60}$$

and

$$P = \frac{\pi^2 EI}{(2L)^2} \tag{61}$$

which is the elastic buckling load for a cantilever.

This method is not the best way to determine the elastic buckling load, but it is an aid in understanding how the series solution for the elastic Q - Δ relation diverges as P approaches P_{el}.

Elastic Buckling as an Eigenvalue Problem

A more familiar approach to the buckling problem is to examine the column in a slightly deflected position and see what load is required to cause an indeterminacy of positions. This procedure is normally carried out using Timoshenko's equations and thus requires the solution of transcendental equations. Using the series equations to set up the problem gives, instead, a polynomial in terms of p. Solving the polynomial for p then gives the elastic buckling load.

Consider the cantilever in a slightly deflected position but under no horizontal load. The only reactive elements at the base will be a vertical force, P, and a resisting moment, P x Δ. Equation (18) yields

$$\Delta = \frac{P\Delta L^2}{EI} S3 \tag{62}$$

or,

$$\Delta(1 - pS3) = 0 \tag{63}$$

There are two possibilities: either Δ is equal to zero or (1 - pS3) is equal to zero. If Δ is equal to zero then the column is not deflected and hence has not buckled. If (1 - pS3) is equal to zero then the deflection is indeterminate; hence, evaluating this expression gives the buckling load.

$$1 - pS3 = 0 \tag{64}$$

or,

$$1 - \frac{p}{2} + \frac{p^2}{24} - \frac{p^3}{720} + \frac{p^4}{40,320} - \ldots = 0 \tag{65}$$

If the first two terms of the series are used one obtains p = 2.0.

Using the first three terms and solving a quadratic yields $p = 2.54$. It can be seen that because of the alternating signs of the series taking an odd number of terms will always overestimate the buckling load, and an even number of terms will always underestimate the buckling load. Figure 18 gives the elastic buckling load versus the number of terms used to compute it from equation (65). These points were calculated by a computer program that solved for the roots of a polynomial by Newton's method. In some cases on other problems, using an odd number of terms gave no real solution. So in general it would be best to use an even number of terms especially since this will underestimate the elastic buckling load. Alternately equation (64) could be solved by trial and error using the tabulated values of S3.

Second-Order Plastic Q - Δ Relation

A first-order plastic analysis of the cantilever beam column would give the horizontal load to cause collapse as MPR/L. MPR would be calculated as outlined in Appendix A. This takes into account the reduction in plastic moment capacity due to axial load but not the $P - \Delta$ effect.

A second-order plastic analysis consists of satisfying statics on the deformed beam column in a mechanism configuration. The plasticity requirement must be satisfied to insure that the assumed mechanism is the correct one. If a plastic hinge is assumed at the base this is sufficient to cause a mechanism. And since the maximum moment is at the base plasticity would not be violated. Summing moments about the base gives

$$Q = \frac{MPR - P\Delta}{L} \tag{66}$$

Thus, for a given P,Q is a decreasing function of Δ. This relation assumes a mechanism and does not apply until the mechanism forms.

Second-Order Elastic-Plastic Behavior

If we assume the cantilever is perfectly elastic until the formation of the hinge, then the second-order elastic Q - Δ relation describes its behavior up to the forming of the mechanism and second-order plastic Q - Δ relation describes the post mechanism behavior. The intersection of the two curves determines the load and deflection at collapse. Figure 19 shows one such intersection for a cantilever. For this simple problem the Δ and Q at collapse could be obtained by solving equations (51) and (66) simultaneously. Giving

$$Q = \frac{MPR}{L} \frac{(1 - pS3)}{(1 - pS4)} \tag{67}$$

and,

$$\Delta = \frac{(MPR)\ L^2}{EI} \frac{(S3 - S4)}{1 - pS4} \tag{68}$$

or,

$$Q = \frac{MPR}{L}\ 1 - \frac{p}{3} - \frac{p^2}{45} - \ldots) \tag{69}$$

and,

$$\Delta = \frac{(MPR)\ L^2}{3EI} (1 + \frac{p}{15} + \frac{2p^2}{315} + \ldots) \tag{70}$$

Letting

$$\frac{MPR}{L} = Q_o, \tag{71}$$

where Q_o is the horizontal load predicted by simple plastic theory, equation (67) gives

$$Q = Q_o \left(1 - \frac{p}{3} - \frac{p^2}{45} - \ldots\right) \tag{72}$$

This shows the reduction in Q below simple theory as a function of p. For p = 1, equation (72) gives

$$Q = 0.644\, Q_o \tag{73}$$

Figure 20 gives the ratio Q/Q_o as a function of p.

Equation (67) along with the moment, thrust equations, from Appendix A, which give MPR are all that are necessary to completely describe the second-order elastic-plastic interaction curve for a cantilever beam column. For a given P, MPR is calculated and then Q is found from equation (67). Elastic buckling is also provided for. When (1 - pS3) is equal to zero, equation (64) gives Q = 0 regardless of the value of MPR/L; and this is exactly the expression found in an eigenvalue approach to elastic buckling. Hence, the elastic buckling load is the value of P that causes equation (67) to be zero and a separate buckling analysis is not needed. If the yield load, T_γ, is less than the elastic buckling load then MPR/L will be zero before (1 - pS3) is equal to zero. For this condition general yielding will control rather than elastic buckling. However, if the elastic buckling load was desired it could be found by putting any non-zero value of MPR/L into equation (64) and finding the p that causes Q to equal zero.

Numerical Techniques

In general it will not be an easy task to obtain a direct formula for Q as a function of P as was possible for this simple case. Hence, several numerical techniques that would give the Q load to cause collapse for any given P load, were tried before going on to more complex problems.

Gauss-Seidel Iteration. (14) The first technique tried was a Gauss-Seidel iteration of equations (50) and (66). While it will generally be easy to obtain the second-order plastic $Q - \Delta$ relation, the second-order elastic $Q - \Delta$ relation will usually be obtained by the recursive iterative approach outlined earlier. Therefore, one would iterate on the second-order elastic curve and use a Gauss-Seidel iteration between the two curves. The procedure is outlined below.

1. Calculate Q from equation (66) assuming $\Delta = 0$.
2. With this Q calculate Δ by the recursive technique from equation (50).
3. Calculate Q from equation (66) using the new Δ.
4. Repeat steps 2 and 3 as required.

This iterative procedure was tried in a program set up for the cantilever and worked fine for values of P well below P_{el}. But as P approached P_{el} the slope of the elastic $Q - \Delta$ curve decreased and the Gauss-Seidel iteration diverged at a load below P_{el}. However, it could be made to work for most cases by changing the technique slightly. If one wrote equation (66) for Δ as a function of Q and equation (50) for Q as a function of Δ then the iterative procedure would work. At this point the technique was abandoned in favor of one which would be easier to apply.

It should be noted that the divergence that occurred was associated with the Gauss-Seidel iterative technique and not the buckling of the column, though the divergence did occur as P increased.

Eliminating Q by Statics. Although the idea of the collapse load being the intersection of the second-order elastic and second-order plastic load deflection curves is important, it is not necessary that the problem be attacked in this specific manner. If a determinate mechanism is assumed on a beam column or a simple frame it will always be possible to eliminate Q from all of the force systems in the structure, thus allowing one to set up an expression for the deflection at the mechanism load independent of Q using the equations of Chapter III. Then Q can be solved for by substituting the Δ so obtained into the plastic Q - Δ relation, in this case, equation (66).

If a plastic hinge is assumed at the base of the cantilever with a plastic moment capacity of MPR, then the shear force at the base can be obtained by summing moments about the base giving the shear force

$$V_{ab} = \frac{MPR - P\Delta}{L} \tag{74}$$

Then equation (18) gives

$$\Delta = \left(\frac{(MPR)L^2}{EI}\right) S3 - \left(\frac{MPR - P\Delta}{1}\right) \frac{L^3}{EI} S4 \tag{75}$$

or

$$\Delta = \frac{(MPR)L^2}{EI} (S3 - S4) + p\Delta S4 \tag{76}$$

Solving for Δ gives

$$\Delta = \frac{(MPR)L^2}{EI} \frac{(S3 - S4)}{1 - pS4} \tag{77}$$

This is not surprising since this is the same solution as previously obtained. But it is important to realize that the only time the deflection is infinite is when $(1 - pS4)$ is zero, which occurs at a value of P much greater than P_{el}. Therefore, the recursive iterative procedure outlined earlier will work on equation (75) even though P equals P_{el}. The buckling phenomenon is not left out, however, for if P is equal to P_{el} then the deflection we obtain from equation (77) will be sufficient to make the $P\Delta$ moment equal MPR, the entire moment capacity of the section, leaving no moment capacity for the horizontal load; thus giving a Q of zero from equation (66). Consequently, an iterative solution is possible which does not diverge when P approaches P_{el} but defines P_{el} as the point where Q is equal to zero. As mentioned before, if general yielding occurs before elastic buckling, then to obtain the elastic buckling load some arbitrary non-zero value of MPR must be used.

A general iterative approach for obtaining the interaction curve of a beam column is

1. Assume a mechanism.

2. Assume a vertical load P.

3. Calculate MPR, using moment-thrust equations from Appendix A.

4. Calculate all forces and moments, assuming $\Delta = 0$, by eliminating Q from force equations.

5. Calculate Δ with these forces using necessary second-order elastic-plastic equations from Chapter III.

6. Recalculate forces.

7. Repeat 5 and 6 until satisfied.

8. Use the Δ so obtained in the $Q - \Delta$ relation for the mechanism to find the Q at collapse.

9. Check plasticity at other possible points of hinges. If plasticity is not violated this is the correct mechanism and one has the Q that would cause collapse with the assumed P. Then increase P and repeat process above until $Q = 0$.

A program was developed that did this for the cantilever and no problems were encountered with divergence.

Inelastic Buckling and the Tangent Modulus Modification of the Second-Order Elastic Plastic Technique

It is well known that a wide flange column of intermediate length will buckle under a compressive load below both the elastic buckling thrust T_{el}, and the yield load, TY, due to the influence of residual stresses. This phenomenon has been adequately expalined by Shanley (15). Shanley gives the critical buckling thrust in the inelastic range T_{in}* as

$$T_{in} = \frac{\pi^2 E_t I}{(sL)^2} \qquad (78)$$

All the terms have been defined with the exception of E_t. E_t is the Tangent Modulus of Elasticity, and it is the slope of the compression stress-strain curve for a short stub column which contains residual stresses.

Equation (78) can be rewritten in terms of the inelastic buckling stress, f_{in}, as follows:

$$f_{in} = \frac{\pi^2 E_t}{(\frac{sL}{r})} \qquad (79)$$

For design purposes the Column Research Council (17) has solved the inelastic problem by giving an equation which expresses the buckling stress as a function of sL/r and is essentially the tangent modulus formula with

*For the cantilever the thrust T equals the applied vertical load P, hence the inelastic buckling thrust T_{in} is identical to the inelastic buckling load P_{in}.

a representative expression for E_t. The AISC manual has adopted this formula and gives the following two formulae to govern buckling:

For $$0 \leq \frac{sL}{r} \leq C_c*$$

$$f_{in} = (1 - \frac{(\frac{sL}{r})^2}{2C_c^2})\ FY** \qquad (80)$$

For $$sL/r \leq C_c$$

$$f_{el} = \frac{\pi^2 E}{(\frac{sL}{r})^2}\ ** \qquad (81)$$

Equations (80) and (81) are plotted in Figure 22 for A33 steel and show how the critical buckling stress varies with the effective length divided by the radius of gyration for a wide flange section.

Sections 1a and 1b of Figure 107 are controlled by inelastic buckling and equation (80). They differ only in that f_{el} in 1a, would be greater than the yield stress, while f_{el} in 1b would be greater than f_{in} but less than FY. Section 2 is controlled by elastic buckling and equation (81).

Suppose one is calculating the interaction curve by the second-order elastic-plastic theory, for a beam column which has an sL/r such that it would buckle in range 1 of Figure 107. If T were equal to T_{in}, Q

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*C_c is the point at which (80) and (81) would give the same value of critical buckling stress and marks the point of switch over from elastic to inelastic buckling.

**A safety factor of one was used since this thesis is concerned with the maximum value of the loads involved.

should equal zero. Second-order elastic-plastic theory would, however, give some value of Q greater than zero since this theory does not give a Q equal to zero until T is equal to T_{el}; and for this sL/r, T_{el} would be above T_{in}. This is to be expected since a second-order elastic-plastic analysis neglects the harmful effect of residual stresses.

Suppose one tries to account for the effect of residual stresses in a semi-empirical manner. One could obtain a Q of zero from equation (67) merely by using T_{el} in evaluating p, S3 and S4 to use in the equation. This is equivalent to multiplying the actual T (T_{in}) by the ratio T_{el}/T_{in}. This is of no help in evaluating T_{in} since its value would have to be known in advance. However, suppose one is calculating the horizontal load the cantilever can support when the axial load T is less than T_{in}. If instead of using the actual T* to evaluate p, S3 and S4 in equation (67) one used instead T', where

$$T' = T\frac{T_{el}}{T_{in}} ** \tag{82}$$

then the value of Q obtained from the equation would be lower than if T had been used and would be nearer the compatibility solution.

Interaction Curves by Various Theories

The results of, (1) simple plastic theory; (2) second-order elastic-plastic theory; and (3) tangent modulus second-order elastic-plastic technique were compared with a more rigorous compatibility

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*T should be used instead of T' in calculating MPR since the reduced moment capacity is not affected by the residual stresses.

**It can be shown that using E_t instead of E, in calculating deflections, E_t being the tangent modulus when Q equals zero and buckling is imminent.

analysis made by Chu. The various interaction curves can be seen in Figure 21. The curves shown in Figure 22 are for a member whose length is such as to cause it to fail by inelastic buckling. Figure 23 shows interaction curves by simple plastic theory, second-order elastic-plastic theory and the tangent modulus second-order elastic-plastic technique for a beam column with an sL/r ratio that would place it in ranges 1a, 1b, and 2 of Figure 22. Examining Figures 21 and 23 shows that

1. Simple plastic theory gives good results only for a very small P load.

2. Second-order elastic-plastic theory gives much better results but is still unconservative particularly as P approaches P_{in}.

3. The tangent modulus second-order elastic-plastic theory gives results very close to the rigorous compatibility analysis by Chu.

Computer Solution

The data for the curves discussed above were obtained from a computer program that, for any cantilever;

1. Calculated the interaction curve by simple plastic theory taking into account the reduction in plastic moment capacity by the equations in Appendix A.

2. Computed the interaction curve by second-order elastic-plastic theory as outlined earlier in this chapter.

3. Found the critical elastic buckling load, P_{el}, by finding the point where Q was equal to zero. If general yielding caused Q to be equal to zero before elastic buckling occurred, an arbitrary value for MPR was put into the equations. Then P_{el} was found as the P which gives a Q of 0.

4. Calculated the inelastic buckling load using equation (94) which is derived as follows:

Equation (81) gives

$$f_{el} = \frac{\pi^2 E}{(\frac{sL}{r})^2} \tag{83}$$

Therefore,

$$(\frac{sL}{r})^2 = \frac{\pi^2 E}{f_{el}} \tag{84}$$

Equation (80) gives

$$f_{in} = (1 - \frac{(\frac{sL}{r})^2}{2C_c{}^2})\ FY \tag{85}$$

Therefore substitution of (84) into (85) gives

$$f_{in} = (1 - \frac{\pi^2 E}{2C_c{}^2 f_{el}})\ FY \tag{86}$$

But

$$TY = A(FY) \tag{87}$$

and

$$f_{el} = \frac{T_{el}}{A} \tag{88}$$

Therefore,

$$f_{in} = FY\ (1 - \frac{\pi^2 E}{2C_c{}^2 FY}\ (\frac{TY}{T_{el}})) \tag{89}$$

And since

$$f_{in} = \frac{T_{in}}{A} \tag{90}$$

$$T_{in} = TY \left(1 - \frac{\pi^2 E}{2C_c^{\,2} FY} \left(\frac{TY}{T_{el}}\right)\right) \tag{91}$$

And since the AISC Manual (8) gives

$$C_c^{\,2} = \frac{2\pi^2 E}{FY} \tag{92}$$

$$T_{in} = TY \left(1 - \frac{TY}{4\,T_{el}}\right) \tag{93}$$

Since for the cantilever the axial thrust T is equal to the applied vertical load P

$$P_{in} = TY \left(1 - \frac{TY}{4P_{el}}\right) \tag{94}$$

If sL/r was greater than C_c then elastic buckling controlled and residual stresses did not affect column buckling; hence, in this case the program printed out "sL/r C_c" and did not compute P_{in} or recompute the interaction curve as below.

5. Otherwise using T_{in} from equation(93) the program recalculated the interaction curve by second-order elastic-plastic theory but replaced T by T' where

$$T' = T\,\frac{T_{el}}{T_{in}} \tag{95}$$

everywhere except in calculating the reduced plastic moment capacity. This gave the tangent-modulus second-order elastic-plastic interaction curve.

CHAPTER V

SWAY FAILURE OF SIMPLE RIGID FRAME

In this chapter, the ideas developed for the cantilever are extended to a simple rigid frame of the type shown in Figure 24. The supports at the base of the structure are not allowed any vertical or horizontal freedom; however, the moment is transmitted from the support to the base of the column through a helical spring. The spring constant, K (moment per unit rotation), may vary anywhere between a fixed end (K equals infinity) and a pinned end (K equals zero). It is assumed that K has a constant value regardless of the value of moment, and that the spring has a greater moment capacity than the column; hence, the M-θ curve for the base will be linear with slope K until the formation of a plastic hinge in the column and horizontal thereafter. To modify the model to include the weak spring problem, where the moment capacity of the spring is below that of the column, would involve only the changing of the plastic moment capacity at the base of the frame from that of the columns to that of the springs.

Failure is restricted to the formation of a sway mechanism as shown in Figure 25-6; hence, no hinges are permitted in the interior of the girder, although vertical loads may be there. To insure that this model gives the true behavior (that is to say the frame would really fail by forming a sway mechanism) plasticity must be checked under each vertical load. This is easily accomplished by statics.

The sway mechanism requires a hinge in all four corners of the

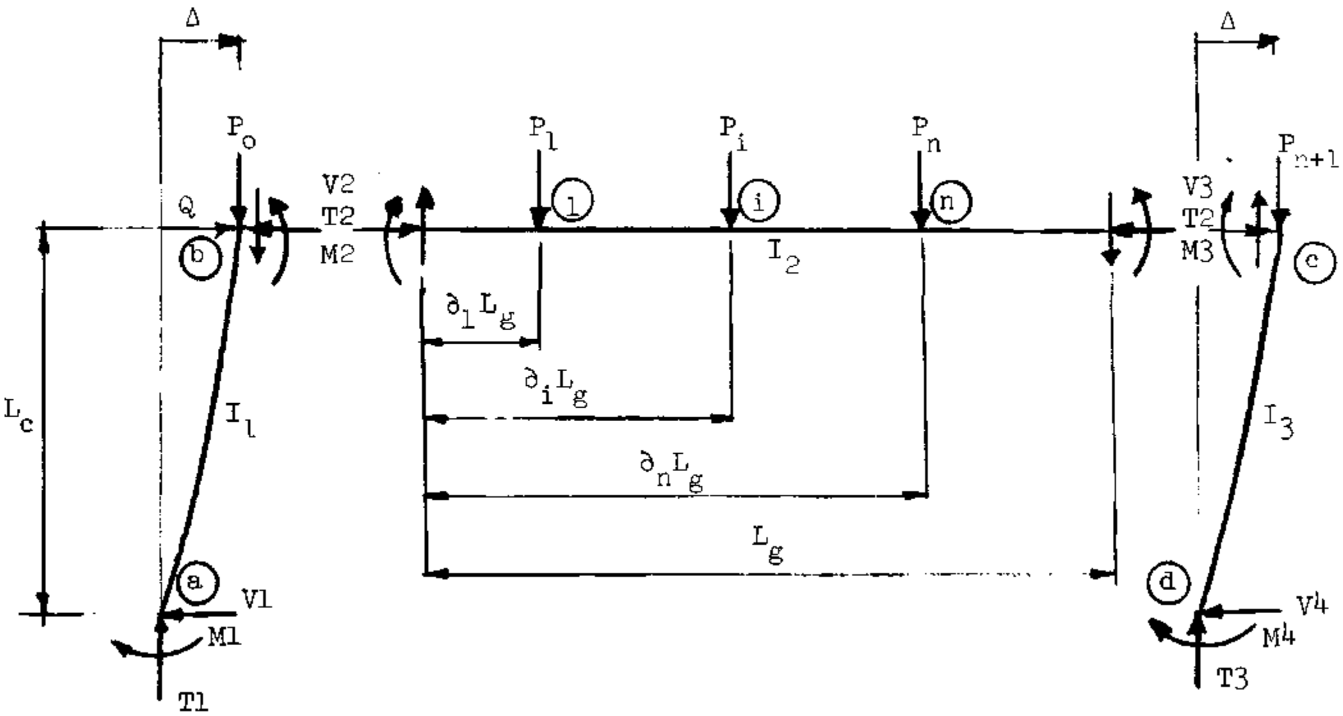

Figure 9. Freebodies for Frame That Fails by a Sway Mechanism

frame; the hinges at the base may be either real hinges or plastic hinges which form in the columns. The hinges that form at the junctions between the columns and the girder may form in either the columns or the girder, depending upon which has the smallest moment capacity, MPR, under that loading.

Second-Order Plastic Equations

It is assumed in this development that the full vertical load is applied, then the horizontal load is applied in increments until a sway mechanism (Figure 25-b) is formed. In the mechanism state M1, M2, M3 and M4 (See Figure 9) are equal to -MPR1, MPR2, -MPR3, and -MPR4 respectively. Summing moments on the girder in Figure 9 gives

$$V3 = -\frac{(MPR2 + MPR3)}{L_g} - \sum_{i=1}^{n} P_i \partial_i \tag{96}$$

and

$$V2 = -\frac{(MPR2 + MPR3)}{L_g} + \sum_{i=1}^{n} P_i (1 - \partial_i) \tag{97}$$

Summing vertical forces on the right hand column of Figure 9 gives

$$T3 = P_{n+1} - V3 \tag{98}$$

or

$$T3 = \sum_{i=1}^{n} P_i \partial_i{}^{*} + \frac{(MPR2 + MPR3)}{L_g} \tag{99}$$

$^{*}\partial_o = 0$ and $\partial_{n+1} = 1$

Similarly for the left column

$$T1 = \sum_{i=0}^{n} P_i \ (1 - \partial_i)^* - \frac{(MPR2 + MPR3)}{L_g} \tag{100}$$

A summation of moments on the left hand column yields

$$V1 = \frac{MPR1 + MPR2 - T1(\Delta)}{L_c} \tag{101}$$

Summing moments on the right column of Figure 9 gives

$$V4 = \frac{MPR3 + MPR4 - T3(\Delta)}{L_c} \tag{102}$$

T2 is equal to V4 and thus

$$T2 = \frac{MPR3 + MPR4 - T3(\Delta)}{L_c} \tag{103}$$

A summation of horizontal forces on the frame in Figure 9 gives

$$Q = V1 + V4 \tag{104}$$

or

$$Q = \frac{MPR1 + MPR2 + MPR3 + MPR4 - (T1 + T3)\Delta}{L_c} \tag{105}$$

Equation (105) is the second-order plastic equation for a sway mechanism. When the proper value for Δ is used (105) will give the horizontal load that will cause collapse. With Δ equal to zero equation (105) gives the simple plastic theory expression for the mechanism load. All of the preceding statics equations hold for the special case of a pin-ended frame

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*$\partial_o = 0$ and $\partial_{n+1} = 1$

provided MPR1 and MPR4 are set equal to zero.

In order for the sway mechanism to form, plasticity must not be violated; that is, the moment in the girder must nowhere exceed the reduced moment capacity of the girder, MPRG. The moment in the girder under any concentrated load, P_i, is given by

$$M_i = MPR2 + V2(\partial_i)L_g - \sum_{j=1}^{i-1} P_i(\partial_i - \partial_j)L_g \qquad (106)$$

For equation (105) to apply, the absolute value of M_i for $1 \le i \le n$ must be less than or equal to MPRG.

To use equation (105), even when Δ is set equal to zero, the relation between the plastic moment capacities and the axial forces in the members must be established. T1, T2, and T3 are functions of the reduced moment capacities, as equations (99), (100), and (103) show. The reduced plastic moment capacities are themselves functions of the axial forces as given by the equations of Appendix A. Since the moment-thrust equations from Appendix A are discontinuous for a wide flange section the only practical approach to their solution is an iterative one:

1. Assume MPR1, MPR2, MPR3, and MPR4 are equal to zero.

2. Calculate T1, T2,* and T3 from equations (99), (100), and (103).

3. Calculate MPR1, MPR2,** MPR3,** and MPR4 using the moment-thrust equations of Appendix A and the axial forces from step 2.

4. Cycle steps 2 and 3 until convergence occurs.

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*It is conservative here to ignore the small reduction in force in T2 due to the deflection. Since T2 is a small force itself, it will not significantly affect the results.

**MPR2 and MPR3 would have to be calculated twice, once assuming that the hinges formed in the columns and once assuming that they formed in the girder. The lowest capacity must be used.

Usually two calculations of the reduced moment capacities are sufficient for slide rule accuracy. In no case did it require more than four cycles on a computer to cause the change in reduced moment capacities to be less than one-tenth of one percent between cycles.

Since the reduced moment capacity is decreasing in the right leg and increasing in the left leg, the sum of the individual reduced moment capacities is essentially constant; hence, one might wonder if it is of any real benefit to compute the exact individual reduced moment capacities MPR1, MPR2, MPR3, and MPR4. It is true that for a sway mechanism the change in individual moment capacities does not significantly change the sum of the moment capacities; but the values of the moment capacities are needed in order to compute the deflection, Δ, at collapse and this does affect the solution. In fact, Bott (9) has shown that in some cases the change in moment capacities actually changes the mechanism that forms.

Having found the axial forces and reduced moment capacities at failure the only quantity needed to evaluate Q from equation (105) is Δ.

Second-Order Elastic-Plastic Equations

The deflection at collapse may be found as follows. Immediately preceding the formation of the last hinge in the mechanism all values of shear, moment, and axial force will be given by the statics equations of the preceding section. In addition continuity will be present at one of the four corners. If it were known which hinge was the last to form then the proper deflection could be calculated. In general it will not be known which is the last hinge to form so the deflection must be calculated assuming the last hinge forms at all four corners.

Case 1: Last Hinge Forms at Base of Left Column

If the last hinge forms at the base of the left column then the slope, shear, and moment are known at this point. Because of the continuity at point a,

$$\theta_{ab} = \frac{MPR1}{K} \tag{107}$$

Equation (97) gives

$$V1 = \frac{MPR1 + MPR2 - T1(\Delta)}{L_c} \tag{108}$$

Since the hinge at "a" is on the verge of forming,

$$M1 = -MPR1 \tag{109}$$

These give all of the forces, moments and slopes necessary to evaluate the deflection from equation (18).

$$\Delta = \Delta_{ba} = \theta_{ab}\, L_c\, S2_{(a,b)} - \frac{(M1)L_c^{\,2}}{EI_1}\, S3_{(a,b)} - \frac{(V1)L_c^{\,3}}{EI_1}\, S4_{(a,b)} \tag{110}$$

Since V1 as given by equation (101) is implicit in Δ, as was the shear force for the cantilever (See equation (74); the same iterative procedure can be used here as was used on the cantilever.

1. Calculate θ_{ab}, V1, and M1 assuming $\Delta = 0$.
2. Calculate Δ from equation (110).
3. Recalculate V1.
4. Cycle steps 2 and 3 until convergence occurs.

This procedure will give the horizontal deflection, Δ, if the last hinge forms at the base of the left column. Naturally if the frame is pinned at this point then a plastic hinge cannot form here; so this possibility need not be considered.

Case II: Last Hinge Forms at the Base of the Right Column

The solution for this case is essentially the same as for Case I.

$$\theta_{dc} = \frac{MPR4}{K} \tag{111}$$

$$V4 = \frac{MPR3 + MPR4 - T3(\Delta)}{L} \tag{112}$$

$$M4 = -MPR4 \tag{113}$$

$$\Delta = \Delta_{cd} = \theta_{dc} \, L_c \, S2_{(c,d)} - \frac{M4L_c^{\,2}}{EI_3} S3_{(c,d)} - \frac{V4L_c^{\,3}}{EI_3} S4_{(c,d)} \tag{114}$$

These equations are solved by the same recursive technique used for equations (107) through (110).

Case III: Last Hinge Forms at the Junction of the Left Column and the Girder

For this case the slope of the column and the girder at point b must be equal until the hinge forms; hence,

$$\theta_{ba} = \theta_{bc} \tag{115}$$

θ_{bc} is given by equation (42) as

$$\theta_{bc} = \frac{M2L_g}{EI_2} SG1_{(b,c)} + \frac{V2L_g^{\,2}}{EI_2} SG2_{(b,c)} - \sum_{i=1}^{n} \frac{P_i(1 - \partial_i)^3 \, L_g^{\,2}}{EI_2} \frac{S4_{(i,c)}}{S2_{(b,c)}} \tag{116}$$

Δ_{ba} is given by equation (37) as

$$\Delta = \Delta_{ba} = \theta_{ba}L_cS11_{(a,b)} + \frac{M1 \; L_c^{\,2}}{EI_1} S12_{(a,b)} + \frac{V1 \; L_c^{\,3}}{EI_1} S13_{(a,b)} \tag{117}$$

Combining equations (115), (116), and (117) with the statics relations for V2, M2, M1, and V1 gives an implicit relation for Δ, as before. Since V1 is the only force that is a function of Δ*, it was assumed that the same sort of iteration procedure as previously used would work.

<u>Numerical Divergence.</u> A computer program was written using this technique and gave good answers for small values of P. But the iterative process failed to converge at a $P<P_{el}$. The reason for this divergence can be seen in the following. The deflection equation for the cantilever (Equation (76)) is of the form

$$\Delta = C + a\Delta \tag{118}$$

where C and "a" are some constants and "a" is always positive. The solution of this equation is

$$\Delta = \frac{C}{1 - a} \tag{119}$$

If "a" equals one then (119) gives an infinite deflection and the iterative solution breaks down. However this does not occur until the applied vertical load is above P_{el}, hence no divergence problems occur in working problems. But the Δ relation obtained from equations (115), (116), and (117) is of the form

$$\Delta = C^{**} - a\Delta \tag{120}$$

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*T2 is also a function of Δ and is needed to evaluate the series parameter for the girder series. This effect was taken into account by iterating on T2 at the same time V1 was iterated on, but it has a very small effect on the solution.

**C is not a true constant since T2 is a function of Δ but it changes in value so little that it may be considered a constant in the operations that follow which cover a small range of Δ.

and divergence occurs at a P less than P_{e1}. The solution to (120) is

$$\Delta = \frac{C}{1 + a} \tag{121}$$

Although equation (121) has a finite solution for any positive value of "a", an iterative approach (similar to the one used for the cantilever) will diverge for an $a \geq 1$. This is a numberical divergence and not associated with the buckling phenomenon.

The problem then arises as to how the basic iterative approach could be retained without involving this sort of divergence. Suppose one tried to solve equation (120) by the iterative approach. One would successively obtain

$$\Delta 1^* = C \tag{122}$$

then

$$\Delta 2 = C - aC \tag{123}$$

or

$$\Delta 2 = C(1 - a) \tag{124}$$

Assuming for the moment that C is always positive, $\Delta 2$ would be less than or equal to zero only if $(1 - a)$ is less than or equal to zero, or stated differently if "a" is greater than or equal to 1. (This is the condition for divergence.) Thus whenever the iterative solution diverges the second estimate of the deflection $\Delta 2$ would be either zero or negative. If the divergence occurs, an approximate value (the true value if C were actually a constant) can be calculated in terms of the first two deflection estimates by equation (127) which can be derived as follows;

*$\Delta 1$ is the first value obtained from the iterative solution and $\Delta 2$ the second.

Combining (124) and (122) gives

$$a = 1 - \frac{\Delta 2}{\Delta 1} \tag{125}$$

Then since C is equal to Δ1, one may combine (121) and (125) to obtain

$$\Delta = \frac{\Delta 1}{1 + 1 - \frac{\Delta 2}{\Delta 1}} \tag{126}$$

or

$$\Delta = \frac{(\Delta 1)^2}{2\Delta 1 - \Delta 2} \tag{127}$$

Thus the basic iterative approach can be maintained and when for some particular vertical load system the second estimate of deflection Δ2 comes out zero or negative, Δ can be calculated from equation (127).

In addition the possibility of C being a negative quantity must be considered. Suppose the hinge at point b is assumed to form last when in the real structure it forms first. The deflection computed assuming the hinge at b formed last would not be the real deflection of the frame. It would however be smaller than the real deflection and could in some cases be negative. This then means that in some cases C could be negative. However this would then give the first estimate of deflection Δ1 equal to a negative quantity. Since in this case the deflection associated with the hinge forming last at b is not needed this deflection could arbitarirly be assigned a zero value, and this is what was done by the computer program developed for this problem.

Case IV: Last Hinge Forms at the Junction of the Right Column and the Girder

The only basic difference between this case and Case III is that continuity remains at point c rather than at point b. Hence,

$$\theta_{cd} = \theta_{cb} \tag{128}$$

Equation (47) gives

$$\theta_{cb} = -\frac{M2L_g}{EI_2} SG11_{(b,c)} - \frac{V2L_g^2}{EI_2} SG12_{(b,c)}$$

$$+ \sum_{i=1}^{n}\left[\frac{P_i(1 - \partial_i)^2 L_g^2}{EI_2}(S3_{(i,c)}\right. \tag{129}$$

$$\left. - (1 - \partial_i)\frac{S4_{(i,c)}S1_{(b,c)}}{})\right]$$

And from equation (37)

$$\Delta = \Delta_{cd} = \theta_{cb} L_c S11_{(c,d)} + \frac{M4L_c^2\ S12_{(c,d)}}{EI_3} + \frac{V4L_c^3\ S13_{(c,d)}}{EI_3} \tag{130}$$

The same test for numerical divergence must be provided here as was used for Case III.

Second-Order Elastic-Plastic Behavior

The largest deflection calculated by the second-order elastic equations will be the deflection at the formation of the sway mechanism. Substituting this deflection into equation (105) gives the value of Q required to form a sway mechanism under the assumed system of vertical loads. The post mechanism behavior is described by equation (105) (except for the stiffening effect of strain hardening). However using this mechanism method does not allow a load-deflection prediction prior to the forming of the mechanism. A conservative estimate of the deflection at any load less than the collapse load may be obtained by assuming the elastic Q - Δ curve is linear between the points of zero load and the mechanism load.

Interaction Curve and Elastic Buckling

An interaction curve of vertical versus horizontal loading may be obtained by calculating the horizontal load capacity for several different values of vertical loading. One selects one of the axial forces as the vertical parameter and then increases all vertical loads in the same ratio as that load. An interaction curve for the pinned-base frame of Figure 26 is shown in Figure 27. For this case only two vertical loads were applied, equal vertical loads, P, at the column tops; hence, either one of these may be the vertical load parameter, the other one being increased the same amount.

The interaction curve of Figure 27 indicates a Q of zero for a P below the elastic buckling load for the frame, as computed by Timoshenko's transcendental* equation for this problem. Even though P differed from the elastic buckling load by only a few percent it was not a numerical error. The value of P that gives a Q of zero by the mechanism method corresponds to point B on the Q - Δ curve shown in Figure 10. Since this value of P is less than the elastic buckling load, no horizontal movement of the frame is possible without the application of a horizontal load; consequently, some horizontal load is required to move the frame from the position of zero horizontal deflection to the horizontal deflection re-

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*Timoshenko's equation can also be derived in series form using the basic series derived herein and an eigenvalue approach. See Appendix D for such a derivation.

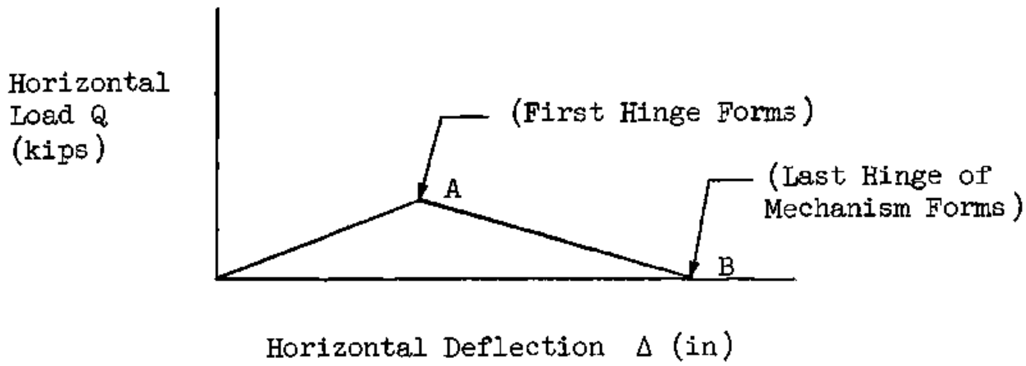

Figure 10. Horizontal Load-Deflection Curve for a Vertical Load less than the Elastic Buckling Load

quired to form a mechanism. It follows, then, that the horizontal load must begin at zero, increase to a maximum (point A in Figure 10) and return to zero at the formation of the mechanism. The only case, then, in which the P giving a zero Q by the mechanism method will agree with the P of elastic buckling theory is one in which the elastic buckling P will satisfy equilibrium for all values of horizontal deflection between zero and collapse (such a case occurs for the cantilever beam-column of Chapter IV); otherwise, the mechanism method P will be less than the elastic buckling P.

Point B in Figure 10 can also be thought of as a point of unstable equilibrium, for a slight additional displacement in the same direction as the original displacement will cause the frame to move into post-mechanism behavior; if, however, the frame is displaced in the opposite direction it will return to the equilibrium position at point B since the predicted vertical collapse load actually required the application of some Q load to reach the incipient mechanism state.

Point of Application of the Q Load

The horizontal load, Q, is shown acting to the right at the top left corner in Figure 24. The statics equations were based on this location. The axial force is given by

$$T2 = \frac{MPR3 + MPR4 - T3(\Delta)}{L_c} \tag{131}$$

If the load were applied in the same direction but at the top right corner, the only difference would be in the magnitude and sense of T2; in this case T2 would be given by

$$T2 = \frac{-(MPR1 + MPR2 - T3(\Delta))}{L_c} \tag{132}$$

Note that now the axial force in the girder will be in tension and the series expressions for θ_{bc} and θ_{cb} will give a value less than the value for zero axial load.

The computer program developed took into account whether the load was applied at the right corner or the left corner of the frame. For the interaction curve of Figure 27 the maximum difference between the values obtained for Q was 0.006 kips, Q being 0.006 kips greater when the load was applied at the right. If the axial force had been assumed to be zero, the Q force would have been somewhere in between the two values found, and would not have significantly affected the accuracy of the solution.

It appears then, that for normal, rectangular, rigid frame structures neglecting the elastic effects of the axial force in the girder would have little effect on the accuracy of the result. Naturally, if a low, very wide frame were being considered, it is possible that the axial load in the girder might appreciably affect the stiffness of the frame, and the effect of the axial load in the girder should be considered.

Comparison With Compatibility Analysis

Adams (3) worked the frame shown in Figure 26 for a P of 0.3PY by both a second-order elastic-plastic analysis and a compatibility analysis. He obtained a Q load of approximately 31 kips by the second-order elastic-plastic analysis whereas the method discussed in this chapter gives a Q load of 30.46 kips. Adams' compatibility analysis gave a Q of 26.2 kips which is 13.0 percent under the load predicted by the methods of this chapter. Simple plastic theory would have yielded a collapse load Q of 42.53 kips which is 40 percent over the second-order elastic solution of 30.46 kips. Figure 11 shows the predicted horizontal loads by the various theories.

It is seen that second-order elastic-plastic theory picks up much of the reduction in horizontal load capacity below that of simple plastic theory. The real behavior of the frame lies somewhere in between the second-order elastic-plastic and the compatibility solutions. Limited experimental studies at Lehigh (2) have shown that second-order elastic-plastic solutions give good correlation to experimental data.

If, however, later studies indicate that the real behavior is closer to the compatibility solution, then the second-order elastic-plastic theory for frames could be modified by using the Tangent Modulus Modification as was done for the cantilever.

Hand Computation

It has already been shown that the effect of the axial load in the girder may be safely ignored for the pinned base frame being studied. The

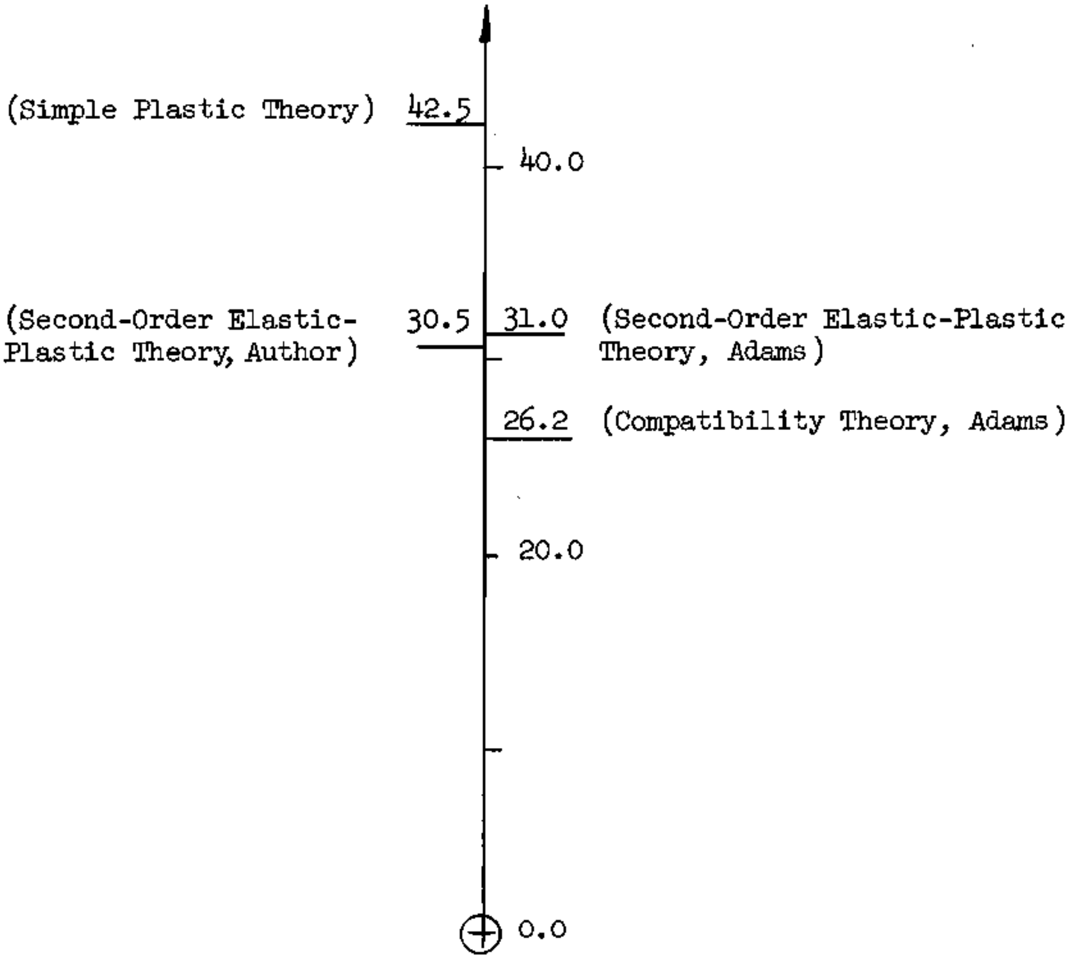

Figure 11. Collapse Load for the Pinned Base Frame of Figure 26 by Various Theories

question then arises as to how much it would affect the computations if the change in the columns' force, due to the shear in the girder, were neglected. If these two assumptions are made it is possible to obtain an expression for Q/Q_o for a pinned base frame as a function of ${}_p$, as was done for the cantilever.

Since the axial force in the girder is assumed equal to zero, θ_{bc}

is given by equation (42) as

$$\theta_{bc} - \frac{(MPR2)\ L_g}{2EI_2} - \frac{(MPR2 + MPR3)}{(L_g)} \frac{L_g^{\ 2}}{6EI_2} \tag{133}$$

and if we assume T1 equals T3, then

$$MPR2 = MPR3 = MPR \tag{134}$$

Therefore,

$$\theta_{bc} = \frac{MPR\ L_g}{6EI_2} \tag{135}$$

With the assumptions made both hinges would form simultaneously so it makes no difference whether we calculate it as in Case III or Case IV. Equation (37) can now be rewritten as

$$\Delta = \Delta_{ba} = \frac{(MPR)\ L_g\ L_c\ S11_{(1,2)}}{6EI_2} + \frac{(MPR - P\Delta)\ L_c^{\ 3}\ S13_{(1,2)}}{L_c\ EI_1} \tag{136}$$

Since T_{ab} is assumed equal to the applied load P the series parameter, p, is given by

$$p = \frac{PL_c^{\ 2}}{EI_1} \tag{137}$$

Since all series are in terms of this parameter the series notation will be dropped. Letting

$$R = \frac{\frac{I_1}{L_c}}{\frac{I_2}{L_g}} \tag{138}$$

and, combining terms, equation (136) gives

$$\Delta = \frac{(MPR)L_c^{\,2}}{EI_1}\left(\frac{R}{6}\right) S11 + S13) - p\Delta S13 \qquad (139)$$

Solving (139) for Δ gives

$$\Delta = \frac{(MPR)L_c^{\,2}}{EI_1}\,\frac{(\frac{R}{6}\,S11 + S13)}{1 + pS13} \qquad (140)$$

Since both shear forces are equal,

$$Q = \frac{2(MPR - P\Delta)}{L} \qquad (141)$$

Substituting in equation (141) for Δ gives

$$Q = \frac{2MPR}{L}\left(1 - \frac{(\frac{R}{6}\,S11 + S13)}{1 + pS13}\right) \qquad (142)$$

or

$$Q = \left(\frac{1 - \frac{pR}{6}\,S11}{1 + pS13}\right) Q_o \qquad (143)$$

Since

$$Q_o = \frac{2MPR}{L} \qquad (144)$$

Changing equation (141) into terms of the basic series and simplifying gives

$$Q = \left(\frac{S1 - \frac{pR}{6}\,S2}{S1 + pS2S3 - pS1S4}\right)Q_o \qquad (145)$$

This equation can be readily solved using the tabulated values of the series.

Figure 28 shows equation (145) plotted as Q/Q_o versus the series parameter p, for various values of R, the column-girder stiffness ratio.

Equation (145) applies for any pinned base frame, such as the one shown in Figure 28. Figure 29 shows a comparison of equation (145) and the computer program which took the change in the axial forces in the column into account. The comparison is for the pinned base frame of Figure 26, which has an R of 1.025. As stated the computer solution was based on the real forces existing in the columns at the formation of the mechanism. However in plotting the results from the program in Figure 29 the parameter p was calculated assuming equal forces in both columns, so that the curves would be comparable.

CHAPTER VI

FAILURE OF SIMPLE FRAMES

In general a frame such as the one shown in Figure 24 may fail by a beam mechanism, a sway mechanism, or a combination mechanism. All three of these mechanisms are shown in Figure 25.

Chapter V dealt with the sway mechanism failure of this frame in detail. In this chapter the techniques necessary to determine the failure load by a combination mechanism are developed. If the effect of the axial force on the girder is neglected then the beam mechanism may be investigated by first-order plastic theory.

P - Δ Effect in the Girder

If the equations of statics are formulated on the frame of Figure 24 in a combination-mechanism configuration (See Figure 25) it is found that the horizontal load, which produces the mechanism, is a function of two deflections. It is a function of the amount the column tops sway, because of the P - Δ effect in the columns. In addition, it is a function of the vertical deflection of the load at the plastic hinges due to the P - Δ effect in the girder. Also, many of the forces necessary to calculate the deflections are functions of both these deflections. Taking the P - Δ effect in the girder into account makes the solution extremely complex, and for the frames studied in this thesis* it has a negligible effect. Neglecting the P - Δ effect in the girder is equivalent to assuming all vertical deflections are zero in the girder, in setting up the statics equations.

Second-Order Plastic Equations

Neglecting the P - Δ effect in the girder, a hinge can occur in the girder only at its ends or under a concentrated load. There will be one possible combination mechanism for each vertical load between the supports. The computer program developed assumed each of these possible mechanisms, calculated the horizontal load to cause it to form, and then checked for plasticity. However it is not necessary to do this. Once a mechanism is assumed, all axial forces** can be calculated and the moment capacities, MPR, checked at all points. When the P - Δ effect is neglected in the girder the moment variation throughout the girder is independent of the horizontal deflection Δ of the column tops; therefore plasticity can be checked by statics in the girder before calculating Δ. The necessary equation to do this will be developed later in this chapter. Checking plasticity in the girder first would then allow the selection of the cor-

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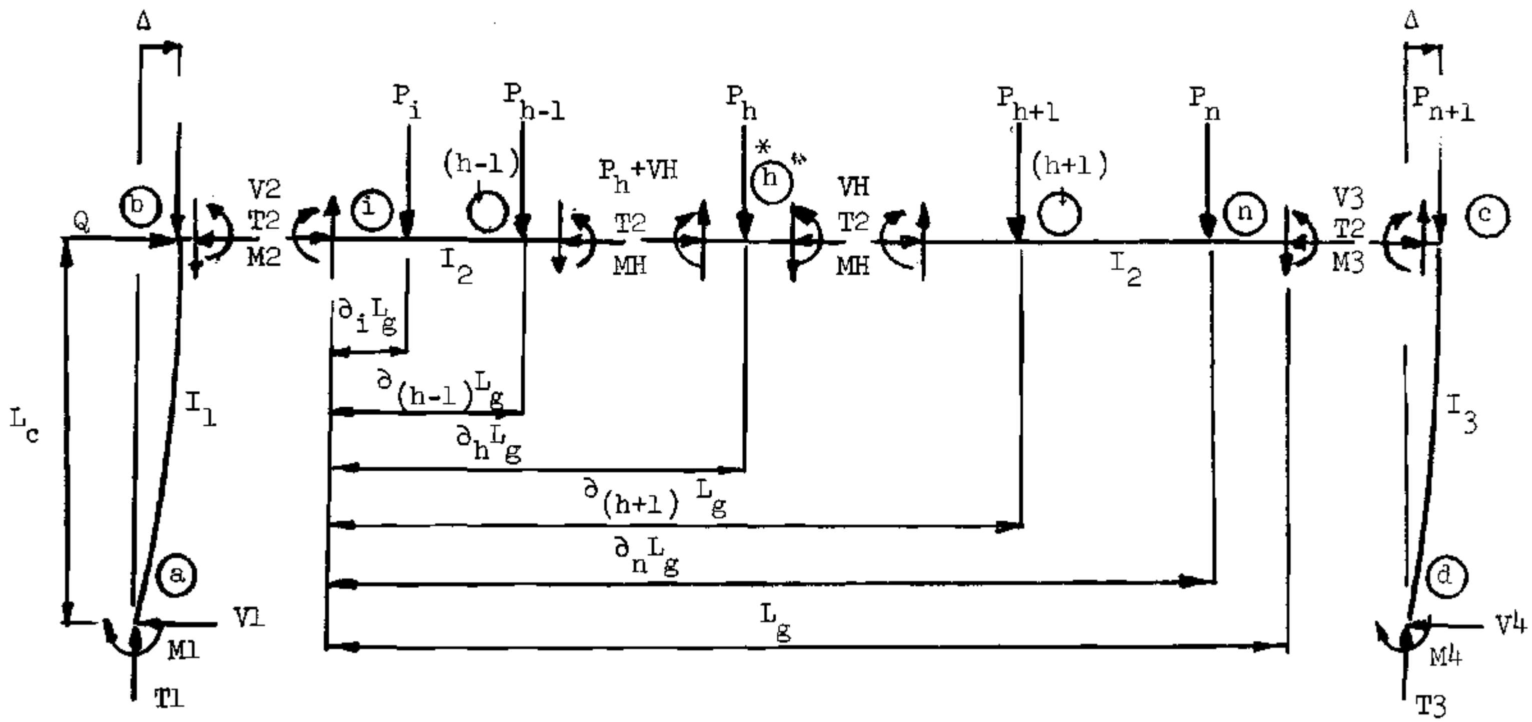
*With reference to the pinned base frame shown in Figure 26 for a P load of 0.3TY the ratio of the total vertical load to the girder's axial load is over 40. Since the P - Δ effect in the columns is proportional to the total vertical load and the P - Δ effect in the girder is proportional to the axial load in the girder, it is clear that neglecting the P - Δ effect in the girder will not appreciably affect the solution. Neglecting the girders' P - Δ effect will be slightly conservative or unconservative depending on whether the horizontal load is applied at the right or left of the frame.

**It is conservative to neglect the effect of the slight reduction in the girding axial force due to deflection in calculating the reduced moment capacity of the girder as was done in Chapter V.



*The hinge in the girder forms at point h under load P_h.

Figure 12. Freebodies for Frame that Fails by a Combination Mechanism

rect mechanism to run the necessary second-order elastic analysis on.

The freebodies of the frame in Figure 12 show the frame in its deformed state neglecting any deflection in the girder. If a sway mechanism is assumed to have formed then M1, MH, M3, and M4 are equal to -MPR1, MPRG, -MPR3, and -MPR4. The hinge in the girder is assumed to have formed under load P_h at point h. The equations to follow will apply for P_h being any one of the n loads between the ends of the girder.

Summing moments on the girder segment (See Figure 12) between the hinge and point c yields

$$V3 = -\frac{(MPR3 + MPRG)}{(1 - \partial_h)\, L_g} - \sum_{i=h+1}^{n} \frac{P_i(\partial_i - \partial_h)}{(1 - \partial_h)} \tag{146}$$

Summing forces vertically on a freebody of the part of the frame to the right of the hinge in Figure 12, yields

$$VH = \sum_{i=h+1}^{n+1} P_i - T3 \tag{147}$$

Summing vertical forces on the right column gives in Figure 12

$$T3 = -V3 + P_{n+1} \tag{148}$$

or

$$T3 = P_{n+1} + \sum_{i=h+1}^{n} \frac{P_i(\partial_i - \partial_h)}{(1 - \partial_h)} + \frac{(MPR3 + MPRG)}{(1 - \partial_h)\, L_g} \tag{149}$$

Then T1 can be found from a summation of forces on the entire frame.

$$T1 = \sum_{i=0}^{n+1} P_i - T3 \tag{150}$$

A summation of moments on the right leg in Figure 12 yields

$$V4 = \frac{(MPR3 + MPR4 - T3(\Delta)}{L_c} \tag{151}$$

T2 is also equal to V4 since the load is applied at the right. Hence,

$$T2 = \frac{MPR3 + MPR4 - T3(\Delta)}{L_c} \tag{152}$$

A summation of moments about the plastic hinge on a freebody of the structure to the left of the hinge in Figure 12 yields

$$V1 = \frac{(MPRG + MPR1 - T1\ (\partial_h L_g + \Delta)}{L_c} + \sum_{i=0}^{h-1} \frac{P_i(\partial_h - \partial_i)\ L_g}{L_c} \tag{153}$$

Summing moments on a free body of the girder between points b and h gives

$$M2 = MPRG - (P_h + VH)\ \partial_h\ L_g - \sum_{i=1}^{h-1} P_i \partial_i L_g \tag{154}$$

Summing vertical forces on the left column yields

$$V2 = T1 - P_o \tag{155}$$

The preceding equations define the force systems in the structure and allow one to check for a violation of plasticity in the girder before performing a deflection analysis. The axial forces, T1, T2, and T3 are functions of the moment capacity of the columns and the girders, hence equations (149), (150), (152), and the moment-thrust equations of

A summation of moments on the right leg in Figure 12 yields

$$V4 = \frac{(MPR3 + MPR4 - T3(\Delta)}{L_c} \tag{151}$$

T2 is also equal to V4 since the load is applied at the right. Hence,

$$T2 = \frac{MPR3 + MPR4 - T3(\Delta)}{L_c} \tag{152}$$

A summation of moments about the plastic hinge on a freebody of the structure to the left of the hinge in Figure 12 yields

$$V1 = \frac{(MPRG + MPR1 - T1\ (\partial_h L_g + \Delta)}{L_c} + \sum_{i=0}^{h-1} \frac{P_i(\partial_h - \partial_i)\ L_g}{L_c} \tag{153}$$

Summing moments on a free body of the girder between points b and h gives

$$M2 = MPRG - (P_h + VH)\ \partial_h\ L_g - \sum_{i=1}^{h-1} P_i \partial_i L_g \tag{154}$$

Summing vertical forces on the left column yields

$$V2 = T1 - P_o \tag{155}$$

The preceding equations define the force systems in the structure and allow one to check for a violation of plasticity in the girder before performing a deflection analysis. The axial forces, T1, T2, and T3 are functions of the moment capacity of the columns and the girders, hence equations (149), (150), (152), and the moment-thrust equations of

Appendix A must be solved by the same process used for similar equations in Chapter V. With the values of moment capacities and axial forces known VH, M2, and V2 can be calculated from equations (147), (154), and (155). Then letting M_i represent the moment, as computed from statics, under any of the concentrated loads, P_i, the moment, M_i, is given by equation (156) as

$$M_i = M2 + V2(\partial_i)\, L_g - \sum_{j=1}^{i-1} P_j(\partial_i - \partial_j) \tag{156}$$

The absolute value of M_i must be less than or equal to the reduced moment capacity at that point (MPRG at all points interior to the girder and MPR2 and MPR3 at the ends; however MPRG could control the value of MPR2 or MPR3, if the girder had a smaller reduced moment capacity than the column).

If a plasticity check of one of the combination mechanisms indicates that plasticity is not violated then the collapse load will be controlled by that mechanism. And the horizontal force, Q, is equal to the sum of the V1 and V4. Hence,

$$\begin{aligned} Q = {} & \frac{(MPR1 + MPR3 + MPR4 + MPRG)}{L_c} \\ & + \sum_{i=0}^{h-1} P_i(\partial_h - \partial_i)\,\frac{L_g}{L_c} \\ & - T1\,\partial_h \frac{L_g}{L_c} - (T1 + T3)\,\frac{\Delta}{L_c} \end{aligned} \tag{157}$$

This is the second-order plastic load deflection relation for a combination mechanism.

Letting $\Delta = 0$ in equation (157) would give the Q by simple plastic theory. To evaluate the Q at collapse a deflection analysis must now be performed on the mechanism using the second-order elastic equations of Chapter III.

Second-Order Elastic Equations

As with the sway mechanism there are four possible places for the hinge to form last for a combination mechanism. If the hinge forms last at the left column base, the hinge in the girder, or the base of the right column, the same deflection equations will be used as were used for cases I, II, and III for the sway mechanism of Chapter V. The only difference is that the force systems to be used in those second-order elastic equations are now given by the statics equations in this chapter instead of the statics equations in the previous chapter. The same iterative procedures were used and no problems of divergence were encountered since the statics equations were of the same form in regard to their dependency on Δ.

If the last hinge should form at the top of the right column a much more complicated deflection analysis is in order. The structure just prior to the last hinge forming is essentially a three hinged frame. (One of the hinges is in the girder so the equations of Chapter III for θ_{bc} and θ_{cb} do not apply.) It is not possible to find any known point from which to start writing the horizontal deflection, Δ, without introducing another slope or deflection as an unknown. A solution can be found for Δ only by enforcing compatibility of deflections at the plastic hinge. While a hinge has formed there the member must still not have torn apart. If it also assumed that the elevation of the ends of the girder do not change, then

$$\Delta_{hb} + \Delta_{ch} = 0 \tag{158}$$

where Δ_{hb} is given in terms of the slope at b, θ_{bc}, by equation (30) as

$$\Delta_{hb} = \theta_{bc}\, \partial_h\, L_g\, S2_{(b,h)} - \frac{M2\, \partial_h{}^2\, L_g{}^2\, S3_{(b,h)}}{EI_2}$$
$$- \frac{V2\, \partial_h{}^3\, L_g{}^3}{EI_2} S4_{(b,h)} + \sum_{i=1}^{h-1} P_i\, \frac{(\partial_h - \partial_i)^3}{EI_2} L_g{}^3 S4_{(i,h)} \tag{159}$$

Equation (37) gives Δ_{ch} in terms of the slope at c, θ_{cb},

$$\Delta_{ch} = \theta_{cb}(1 - \partial_h)\, L_g\, S11_{(h,c)}$$
$$+ \frac{MH(1 - \partial_h)^2 L_g{}^2}{EI_2} S12_{(h,c)}$$
$$+ \frac{VH(1 - \partial_h)^3 L_g{}^3}{EI_2} S13_{(h,c)} \tag{160}$$
$$- \sum_{i=h+1}^{n} \frac{P_i(1 - \partial_i)^3 L_g{}^3}{EI_2} \left(S11_{(h,c)} S3_{(i,c)} - S4_{(i,c)}\right)$$

Substituting (159) and (160) into (158) gives

$$(\partial_h L_g S2_{(b,h)})\, \theta_{bc} + ((1 - \partial_h) L_g\, S11_{(h,c)})\, \theta_{cb}$$
$$= \frac{M2\, \partial_h{}^2 L_g{}^2}{EI_2} S3_{(b,h)} + \frac{V2\, \partial_h{}^3 L_g{}^3}{EI_2} S4_{(b,h)} \tag{161}$$
$$- \frac{MH(1 - \partial_h)^2 L_g{}^2}{EI_2} S12_{(h,c)} - \frac{VH(1 - \partial_h)^3 L_g{}^3}{EI_2} S13_{(b,c)}$$

$$- \sum_{i=1}^{h-1} \frac{P_i(\partial_h - \partial_i)^3 L_g^{\ 3}}{EI_2} S4_{(i,h)}$$

$$+ \sum_{i=h+1}^{n} \frac{P_i(1 - \partial_i)^3 L_g^{\ 3}}{EI_2} (S11_{(h,c)} S3_{(i,c)} - S4_{(i,c)}) \tag{161}$$

This gives the relation between the slopes at the ends of the girders. Because of the continuity at these points,

$$\theta_{bc} = \theta_{ba} \tag{162}$$

and

$$\theta_{cb} = \theta_{cd} \tag{163}$$

Also,

$$\Delta_{ba} = \Delta_{cd} = \Delta \tag{164}$$

Hence we can write two equations for Δ in terms of θ_{bc} and θ_{cb}. Combined with equation (161) this will give three equations and three unknowns, Δ, θ_{bc} and θ_{cb}, which can then be solved. On the left column from equation (37)

$$\Delta = \Delta_{ba} = \theta_{bc} L_c S11_{(a,b)} + \frac{M1L_c^{\ 2}}{EI_1} S12_{(a,b)} \tag{165}$$

$$+ \frac{V1L_c^{\ 3}}{EI_1} S13_{(a,b)}$$

On the right column from equation (37)

$$\Delta = \Delta_{cd} = \theta_{cd} L_c S11_{(c,d)} + \frac{M4L_c^{\ 2}}{EI_1} S12_{(c,d)} \tag{166}$$

$$+ \frac{V4L_c^{\ 3}}{EI_1} S13_{(c,d)} \tag{166a}$$

Equations (161), (165) and (166) can be solved for Δ by Cramers rule giving

$$\Delta = \Big[(\partial_h L_g\ S2_{(b,h)}) \Big(\frac{M1L_c^{\ 2}}{EI_1} S12_{(a,b)} +$$

$$\frac{V1L_c^{\ 3}}{EI_1} S13_{(a,b)}\Big) (L_c\ S11_{(c,d)})$$

$$+ ((1 - \partial_h)\ L_g S11_{(h,c)})(L_c S11_{(a,b)})\Big(\frac{M4L_c^{\ 2}}{EI_1} S12_{(c,d)} +$$

$$\frac{V4L_c^{\ 3}}{EI_1} S13_{(c,d)}\Big) \tag{167}$$

$$+ \Big((L_c S11_{(c,d)})(L_c S11_{(a,b)})\Big(\frac{M2\ \partial_h^{\ 2} L_g^{\ 2}}{EI_2} S3_{(b,h)} +$$

$$\frac{V2\ \partial_h^{\ 3} L_g^{\ 3}}{EI_2} S4_{(b,h)} - \frac{MH(1 - \partial_h)^2 L_g^{\ 2}}{EI_2} S12_{(h,c)} -$$

$$\frac{VH(1 - \partial_h)^3 L_g^{\ 3}}{EI_2} S13_{(h,c)} - \sum_{i=1}^{h-1} \frac{P_i(\partial_h - \partial_i)^3 L_g^{\ 3}}{EI_2} S4_{(i,h)} +$$

$$\sum_{i=h+1}^{n} \frac{P_i(1-\partial_i)^3 L_g^{\ 3}}{EI_2} (S11_{(h,c)} S3_{(i,c)} - S4_{(i,c)})\Big)\Big]$$

$$\div \left[(\partial_h L_g S2_{(b,h)})(L_c S11_{(c,d)}) + ((1 - \partial_h)L_g S11_{(h,c)})(L_c S11_{(a,b)})\right] \quad (167a)$$

Equation (167) is of the same form as previous deflection equations, implicit in Δ, and hence is solved by the iterative approach used earlier. V1 and V4 are the only forces which are functions of Δ, and since they are both decreasing functions of Δ (See equations (151) and (153)) and have a positive sign on the right side of equation (167), equation (167) will be of the form

$$\Delta = C - a\Delta \quad (168)$$

This is the same type equation as occurred in Chapter V for the last hinge forming at the left or right end of the girder and hence the same test for numerical divergence applies.

Comparison with the Analytical and Experimental Results at Lehigh

The rigid frame shown in Figure 30 was analyzed by second-order elastic-plastic theory at Lehigh, and then tested experimentally there (2). The frame was tested by applying the full vertical load first. This gave a load in each column of approximately 0.26 TY. The horizontal load Q was then applied and the experimental horizontal load-deflection curve obtained. The experimental results were compared with an analytical study by second-order elastic-plastic theory. The method by which this analysis was performed consisted of a second-order elastic analysis between successive hinges that formed. Plasticity was concentrated at the locations of the hinges. The second-order elastic analyses were performed using slope deflection equations modified to account for the reduction in stiffness of the frame.

Lehigh completely ignored the axial force in the girder and the change in axial forces in the columns. The measured and predicted maximum horizontal loads were in close agreement, both being approximately 16.9 kips.

An analysis run using second-order elastic-plastic theory and the mechanism method as outlined in Chapters V and VI gave a horizontal Q of 17.0 kips. This analysis took into account the changing axial forces in the columns and the axial forces in the girder. (The affect on the stiffness of the frame of the axial force in the girder was accounted for, but its statical P-Δ effect was neglected.

The failure mechanism was a combination mechanism with hinges at points a, l, c and h. The last hinge to form was the hinge at a. This agreed with the mechanism and last hinge found by analysis and observation at Lehigh.

An analysis by simple plastic theory would give a collapse load of 20.3 kips. This is almost 20 per cent higher than the collapse load observed and predicted by second-order elastic-plastic theory.

Base Rotations and the Hinge Shift

The Lehigh frame was designed to be nominally fixed-base; however, some rotation was observed at the base of the structure and this was taken into account by the spring constant, K (K was obtained experimentally). It has been shown (18) that when two members are framed together the hinge will form at a distance away from the face of the joint equal to the depth of the member in which the hinge forms. This was accounted for by reducing the length of the columns* in the analysis.

Point of Application of the Horizontal Load

If Q is applied at the left corner of the frame as shown in Figure 24 the axial force in the girder will be a compressive force equal to V4. If the Q force is applied in the same direction but at the top left corner then the axial force in the girder will be a tensile force equal to V1. The computer program developed, analysed the frame of figure 24 for the load applied at the right and at the left. The defference in horizontal load capacity was only 0.042 kips less than two and one-half percent of the maximum load. If zero axial force had been assumed in the girder the calculated Q would have been between the two values found and would not have caused an appreciable error.

Frame Solution

The technique developed to calculate the horizontal collapse load for the frame of Figure 24 under any given system of vertical concentrated loads is as follows:

1. Select the sway mechanism or one of the combination mechanisms which causes collapse by checking for a violation of plasticity.
    a) This will require an iterative solution of the relation between the reduced moment capacities and the axial forces.
    b) The mechanism that does not violate plasticity is the correct one. If all of the mechanisms violate plasticity then a beam mechanism controls and may be analysed by first-order theory.

---

*The plastic hinge formed in the column rather than in the girder for this case.

2. Calculate the deflection of the column tops, Δ, assuming in turn that the last hinge in the mechanism forms at all possible places.

   a) The greatest deflection so computed is the deflection of the frame at the formation of the mechanism.

   b) The various calculations of Δ are performed using the second-order elastic equations derived in Chapters V and VI. The values of the forces and moments to be used in these equations are given by the equations of statics also found in these chapters.

      1) Since some of the forces in the statics equations are functions of Δ an iterative solution to the second-order elastic solution is used. First, assume Δ is equal to zero and calculate the forces and moments required. Second, calculate the deflection Δ with these forces. Recalculate the forces necessary then recalculate Δ. This process is continued until convergence occurs.*

3. Substitute this deflection into the appropriate second-order plastic load-deflection relation. This will then give the horizontal load to cause collapse.

4. Post mechanism load deflection behavior may be predicted by assuming the second-order plastic Q - Δ equation to apply. A conservative deflection estimate of pre-mechanism deflections may be made assuming the second-order elastic load deflection curve is linear.

---

*A numerical divergence sometimes occurs. This possibility is discussed in detail in Chapter V.

CHAPTER VII

CONCLUSIONS

The second-order elastic equations derived in Chapter III can be combined with statics, compatibility and plasticity such that the collapse load and deflection for beam columns and simple frames can be predicted by second-order elastic-plastic theory.

The accuracy of a second-order elastic-plastic solution was not appreciably affected by ignoring the axial forces in the girders of the simple frames studied.

Ignoring the change in axial forces in the columns of simple frames due to the horizontal load and displacement is unconservative.

Second-order elastic-plastic theory gives a good prediction of the collapse load for simple frames when compared with limited experimental results, but is slightly unconservative when compared with a rigorous compatibility theory.

Part of the small difference between the second-order elastic-plastic theory and the rigorous compatibility theory may be picked up by a simple empirical technique, which involves using the tangent modulus in second-order elastic-plastic theory.

The elastic buckling load, or a conservative estimate of it may be obtained as a point on the interaction curve of vertical and horizontal loading and does not require a separate buckling analysis.

The use of the series in the second-order elastic-deflection equations gives a clear insight into the structural behavior of beam columns and simple frames.

The series equations can be used for tensile axial loads as well as for compressive axial loads, and for zero axial load the series give the same results as first order elastic theory.

CHAPTER VIII

RECOMMENDATIONS

It is felt that the ideas developed in this thesis could be extended to multistory, multibay frames. The chief difficulty to such solutions being partial collapse, i.e., mechanisms which leave a portion of the structure indeterminate.

The author feels that a comprehensive test program is in order to determine how accurately second-order elastic-plastic behavior predicts the behavior of frames over the entire loading range. (The complete interaction curve should be obtained experimentally and compared with an interaction curve obtained by second-order elastic-plastic theory.)

It is felt that the elastic deflection equations developed in Chapter III will have application to other problems such as the buckling of non-uniform columns and beam-columns on an elastic foundation.

APPENDIX A

REDUCTION IN MOMENT CAPACITY DUE TO AXIAL FORCE

If a short segment of a wide flange as shown in Figure 13 is subjected to an axial load T less than TY and then has an increasing moment M applied, the progressive stress distribution over the depth of the wide flange would look as shown in Figure 14, stress distribution a through d. The moment capacity of stress distribution e is not significantly different from the moment capacity of stress distribution d and since it is much easier to calculate it has been used as the mathematical model for obtaining the maximum value of moment, MPR, the section can develop with the axial load T acting. For any value of T the value of MPR will be less than MP, the plastic moment capacity, if no axial force is acting. It is realized, of course, that the stress distribution shown in e is impossible as it would require an infinite curvature.

Second-Order Elastic-Plastic Theory assumes that the M - ϕ relation is linear all the way up to the formation of the plastic hinge at the reduced plastic moment capacity, MPR, and that the relation is given by the normal elastic flexural relation, $\phi = M/EI$. Beyond this point the section is assumed to deform freely under constant load. M - ϕ curves for various values of T for an 8WF31 are shown in Figure 15. They are shown for both the Second-Order Elastic-Plastic Theory and the more rigorous Compatibility Theory.

Reference (11) gives the equations necessary to describe the relation between MPR, MP, T and the section properties. They are readily

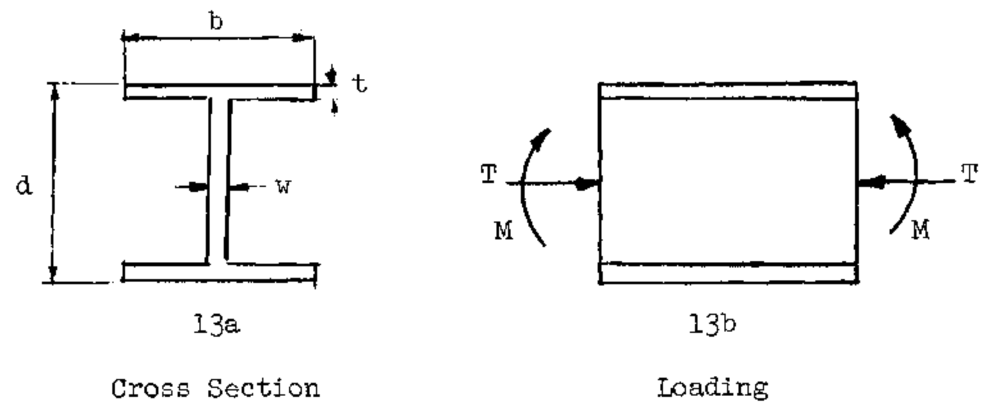


Figure 13. Short Segment of a Wide Flange Beam-Column

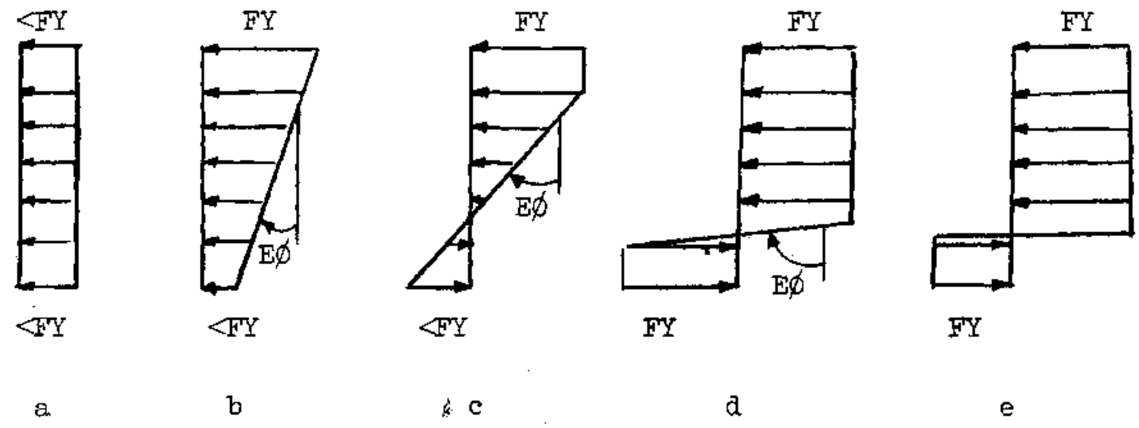


Figure 14. Progressive Stress Distribution Over the Depth of a Wide Flange Beam-Column

derivable from a consideration of equilibrium of the beam column segment with a stress distribution as shown in Figure (14-e). They are, for

$$0 \leq T \leq FY\ w\ (d - 2t)$$

$$MPR = MP - \frac{T^2}{4FY} \qquad \text{(A-1)}$$

and for

$$FY\ w\ (d - 2t) \leq T \leq TY$$

$$MPR = \frac{FY}{2}\left(d\left(A - \frac{T}{FY}\right) - \frac{1}{2b}\left(A - \frac{T}{FY}\right)^2\right) \qquad \text{(A-2)}$$

Two equations are required because of the discontinuity at the junction of the web and the flange. The first equation applies as long as the bottom flange is completely in tension and the second equation is used when there is some compressive stress in the bottom flange.

These equations were programmed to give the MPR-T relation for any wide flange section. This program was included as a procedure in the larger programs that calculated the collapse load and deflections for simple frames.

The AISC manual gives two equations that give essentially the same results for a typical wide flange section. They are around four percent unconservative for P/PY of 0.15. Although the more exact equations from the previous page were used in all work herein, the AISC equations are listed for comparison.

$$0 \leq T \leq 0.15TY$$

$$MPR = MP \qquad \text{(A-3)}$$

and for

$$0.15P \leq T \leq TY$$

$$MPR = MP\left(1.18 - 1.18\left(\frac{T}{TY}\right)\right) \tag{A-4}$$

APPENDIX B

TYPICAL SERIES OPERATIONS

In order to get expressions in terms of the parameter p for series expressions it was necessary to make certain operations with the basic series. In one case it was desired to expand the expression:

$$\frac{S3 - S4}{1 - pS3}$$

in terms of p.

This was accomplished as follows:

$$S3 = \frac{1}{2} - \frac{p}{24} + \frac{p^2}{720} - \frac{p^3}{40,320} + \ldots$$

$$- S4 = - \frac{1}{6} + \frac{p}{120} - \frac{p^2}{5040} + \frac{p^4}{362,880} - \ldots$$

$$= \quad \frac{1}{3} - \frac{p}{30} + \frac{p^2}{840} - \frac{p^3}{45,360}$$

and

$$1 - pS3 = 1 - \frac{p}{2} + \frac{p^2}{24} - \frac{p^3}{720} + \ldots$$

$$\frac{S3 - S4}{1 - pS3} = \frac{\frac{1}{3} - \frac{p}{30} + \frac{p^2}{840} - \frac{p^3}{45,360} + \ldots}{1 - \frac{p}{2} + \frac{p^2}{24} - \frac{p^3}{720} + \ldots}$$

$$
\begin{array}{r|l}
 & \frac{1}{3} + \frac{2p}{15} + \frac{17p^2}{315} + \frac{62p^3}{2835} + \cdots \\
\hline
1 - \frac{p}{2} + \frac{p^2}{24} - \frac{p^3}{720} & \frac{1}{3} - \frac{p}{30} + \frac{p^2}{840} - \frac{p^3}{45,360} \\
 & \\
 & \frac{1}{3} - \frac{p}{6} + \frac{p^2}{72} - \frac{p^3}{2160} \\
 & - \quad + \quad - \quad + \\
\cline{2-2}
 & 0 + \frac{2p}{15} - \frac{4p^2}{315} + \frac{p^3}{2268} \\
 & \quad + \frac{2p}{15} - \frac{p^2}{15} + \frac{p^3}{180} \\
 & \quad - \quad + \quad - \\
\cline{2-2}
 & \quad 0 + \frac{17p^2}{315} - \frac{89p^3}{5670} \\
 & \quad\quad + \frac{17p^2}{315} - \frac{17p^3}{630} \\
 & \quad\quad - \quad + \\
\cline{2-2}
 & \quad\quad\quad 0 + \frac{62p^3}{2835} \\
 & \quad\quad\quad\quad + \frac{62p^3}{2835} \\
 & \quad\quad\quad\quad - \\
\cline{2-2}
 & \quad\quad\quad\quad 0
\end{array}
$$

$$\frac{S3 - S4}{1 - pS4} = \frac{1}{3} + \frac{2p}{15} + \frac{17p^2}{315} + \frac{62p^3}{2835} + \cdots$$

The same type operations were performed to obtain the series expansions in Appendix C.

APPENDIX C

SERIES EXPANSIONS*

$$S11_{(a,b)} = \frac{S2_{(a,b)}}{S1_{(a,b)}} = 1 + \frac{p}{3} + \frac{2p^2}{15} \qquad \text{(C-1)}$$

$$S12_{(a,b)} = \frac{S2_{(a,b)}}{S1_{(a,b)}} S2_{(a,b)} - S3_{(a,b)}$$

$$= \frac{1}{2} + \frac{5p}{24} + \frac{61p^2}{720} \qquad \text{(C-2)}$$

$$S13_{(a,b)} = \frac{S2_{(a,b)}}{S1_{(a,b)}} S3_{(a,b)} - S4_{(a,b)}$$

$$= \frac{1}{3} + \frac{2p}{15} + \frac{17p^2}{315} \qquad \text{(C-3)}$$

$$S14_{(a,b)} = \frac{S2_{(a,b)}}{S1_{(a,b)}} S4_{(a,b)} - S5_{(a,b)}$$

$$= \frac{1}{8} + \frac{7p}{144} + \frac{791p^2}{40,320} \qquad \text{(C-4)}$$

$$SG1_{(a,b)} = \frac{S3_{(a,b)}}{S2_{(a,b)}} = \frac{1}{2} + \frac{p}{24} + \frac{p^2}{240} \qquad \text{(C-5)}$$

---

*In all expressions in this appendix, $p = \dfrac{T_{ab} L_{ab}^2}{EI}$

$$SG2_{(a,b)} = \frac{S4_{(a,b)}}{S2_{(a,b)}} = \frac{1}{6} + \frac{7p}{360} + \frac{31p^2}{15,120} \tag{C-6}$$

$$SG3_{(a,b)} = \frac{S5_{(a,b)}}{S2_{(a,b)}} = \frac{1}{24} + \frac{p}{180} + \frac{73p^2}{120,960} \tag{C-7}$$

$$\begin{aligned} SG11_{(a,b)} &= S2_{(a,b)} - S3_{(a,b)} \frac{S1_{(a,b)}}{S2_{(a,b)}} \\ &= \frac{1}{2} + \frac{p}{24} + \frac{p^2}{240} \end{aligned} \tag{C-8}$$

$$\begin{aligned} SG12_{(a,b)} &= S3_{(a,b)} - S4_{(a,b)} \frac{S1_{(a,b)}}{S2_{(a,b)}} \\ &= \frac{1}{3} + \frac{p}{45} + \frac{2p^2}{945} \end{aligned} \tag{C-9}$$

$$\begin{aligned} SG13_{(a,b)} &= S4_{(a,b)} - S5_{(a,b)} \frac{S1_{(a,b)}}{S2_{(a,b)}} \\ &= \frac{1}{8} + \frac{p}{144} + \frac{77p^2}{120,960} \end{aligned} \tag{C-10}$$

The series expressions listed in this Appendix were obtained by the formal methods outlined in Appendix B, hence they may not converge for all values of p that are physically relevant. In such cases, values may be obtained by calculating the basic series and combining them arithmetically.

APPENDIX D

EIGENVALUE APPROACH TO ELASTIC BUCKLING OF PINNED BASE FRAME

Timoshenko (5) has given the following transcendental equations to determine the elastic buckling load for a pinned base frame such as the one shown in Figure 26.

$$kL_c \tan(kL_c) = \frac{6}{R} \tag{D-1}$$

where

$$P_{ei} = k^2\, EI_1 = T_{el} \tag{D-2}$$

(D - 1) may be expanded in series form as follows:

$$kL_c(kL_c + \frac{(kL_c)^3}{3} \; \frac{2(kL_c)^5}{15} + \ldots) = \frac{6}{R} \tag{D-3}$$

However,

$$k^2 L_c^{\,2} = \frac{PL_c^{\,2}}{EI} = p \tag{D-4}$$

Therefore,

$$p + \frac{p^2}{3} + \frac{2p^3}{15} + \ldots = \frac{6}{R} \tag{D-5}$$

Solving for p from equation (D-5) would give the p at buckling and then

$$P_{el} = p\,\frac{EI_1}{L_c^{\,2}} \tag{D-6}$$

Equation (D-5) can be obtained easily by using an eigenvalue approach and the series techniques developed herein.

Consider the frame is a slightly deflected shape under zero horizontal load. The axial force in the girder is equal to zero; hence from equation (35).

$$\theta_{bc} = \frac{M_{bc} L_g}{2EI_2} + \frac{(M_{cb} - M_{bc})\ L_g}{6EI_2} \tag{D-7}$$

but since there is no shear force in the columns

$$M_{bc} = P\Delta_{ba} \tag{D-8}$$

and

$$M_{cb} = -P\Delta_{cd} \tag{D-9}$$

Therefore recognizing that Δ_{cd} equals Δ_{ba}

$$\theta_{bc} = \frac{P\Delta_{ba} L_g}{6EI_2} \tag{D-10}$$

And since from equation (34)

$$\Delta_{ba} = \theta_{bc}\ L_c\ S11_{(a,b)} \tag{D-11}$$

or

$$\Delta_{ba} = \frac{P\Delta_{ba}\ L_g\ L_c}{6EI_2}\ S11_{(a,b)} \tag{D-12}$$

or

$$\Delta_{ba}\ (1 - \frac{p\ R\ S11_{(a,b)}}{6}) = 0 \tag{D-13}$$

Either $\Delta_{ba} = 0$, which is a trivial solution, or else

$$\left(1 - \frac{p\ R\ S11_{(a,b)}}{6}\right) = 0 \tag{D-14}$$

which may be written in terms of p as

$$\frac{6}{R} = p + \frac{p^2}{3} + \frac{2p^3}{15} + \ .\ .\ . \tag{D-15}$$

which is identical to a series expansion of Timoshenko's transcendental equation.

For small values of the column - girder stiffness ratio R, equation D-15 will not give a good approximation to the elastic buckling load. The simplest solution is to rewrite the equation in terms of S1 and S2 and solve it by trial and error using the tabulated values of S1 and S2.

APPENDIX E

BASIC SERIES TABULATED

P	S1	S2	S3	S4	S5
0.0	1.000	1.000	0.500	0.1667	0.0416
0.2	0.902	0.967	0.492	0.1650	0.0414
0.4	0.807	0.935	0.484	0.1634	0.0411
0.6	0.715	0.903	0.475	0.1617	0.0408
0.8	0.626	0.872	0.467	0.1601	0.0406
1.0	0.540	0.842	0.460	0.1585	0.0403
1.2	0.458	0.812	0.451	0.1569	0.0400
1.4	0.378	0.783	0.444	0.1554	0.0398
1.6	0.301	0.754	0.437	0.1538	0.0395
1.8	0.227	0.726	0.429	0.1523	0.0392
2.0	0.156	0.699	0.422	0.1508	0.0390
2.2	0.087	0.671	0.414	0.1493	0.0387
2.4	0.022	0.645	0.408	0.1478	0.0385
2.5	-0.010	0.632	0.404	0.1470	0.0383
3.0	-0.161	0.570	0.387	0.1434	0.0377
3.5	-0.296	0.511	0.370	0.1398	0.0371
4.0	-0.416	0.455	0.354	0.1363	0.0365
4.5	-0.523	0.402	0.339	0.1329	0.0359
5.0	-0.617	0.352	0.323	0.1296	0.0353
5.5	-0.699	0.305	0.309	0.1264	0.0347
6.0	-0.770	0.261	0.295	0.1232	0.0342
6.5	-0.830	0.219	0.282	0.1202	0.0336
7.0	-0.878	0.180	0.269	0.1172	0.0331
7.5	-0.920	0.143	0.256	0.1142	0.0325
8.0	-0.951	0.109	0.244	0.1114	0.0320
8.5	-0.975	0.077	0.232	0.1086	0.0315
9.0	-0.990	0.047	0.221	0.1059	0.0310
9.5	-0.998	0.019	0.210	0.1032	0.0305
10.0	-1.000	-0.007	0.200	0.1000	0.0300

APPENDIX F

FIGURES 15 THROUGH 30

The solid M - ϕ curves are those from reference 13. They assume a typical residual stress distribution. The dashed lines represent the linear M - ϕ curves assumed by second-order elastic-plastic theory. M_y and ϕ_y are the moment and curvature which would cause the outer fibers of the section to yield with no residual stresses and no axial load acting

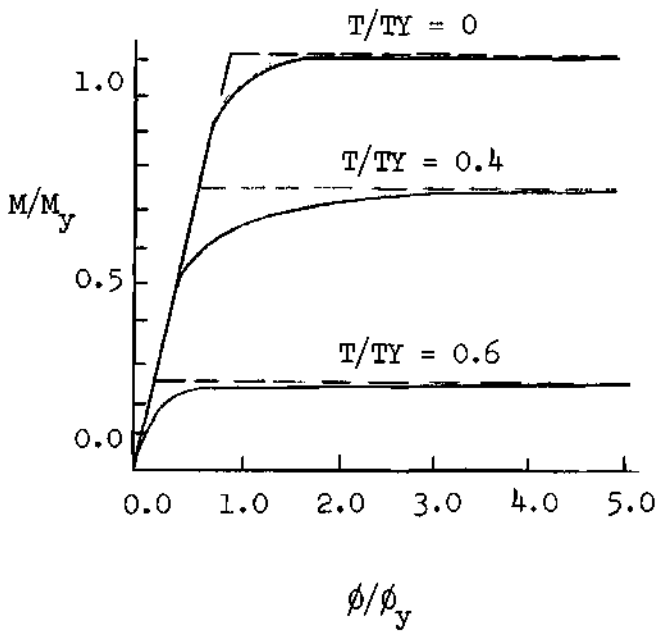
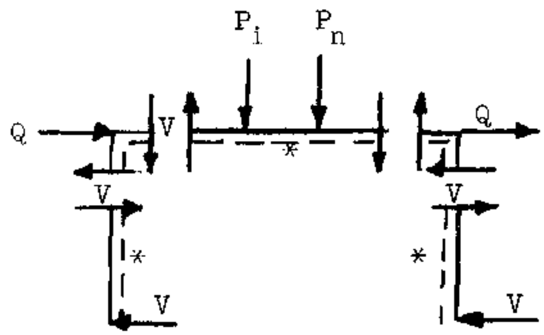
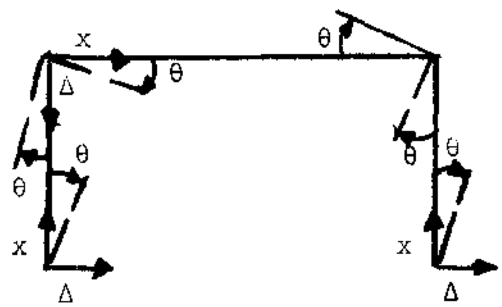


Figure 15. Dimensionless M - ϕ Curves for an 8 WF 31



16a

Positive Shear, Moment, and Loading



16b

Positive Coordinates and Slopes

Figure 16. Sign Convention for a Frame

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*Positive moment produces tension on the dashed sign of frame.

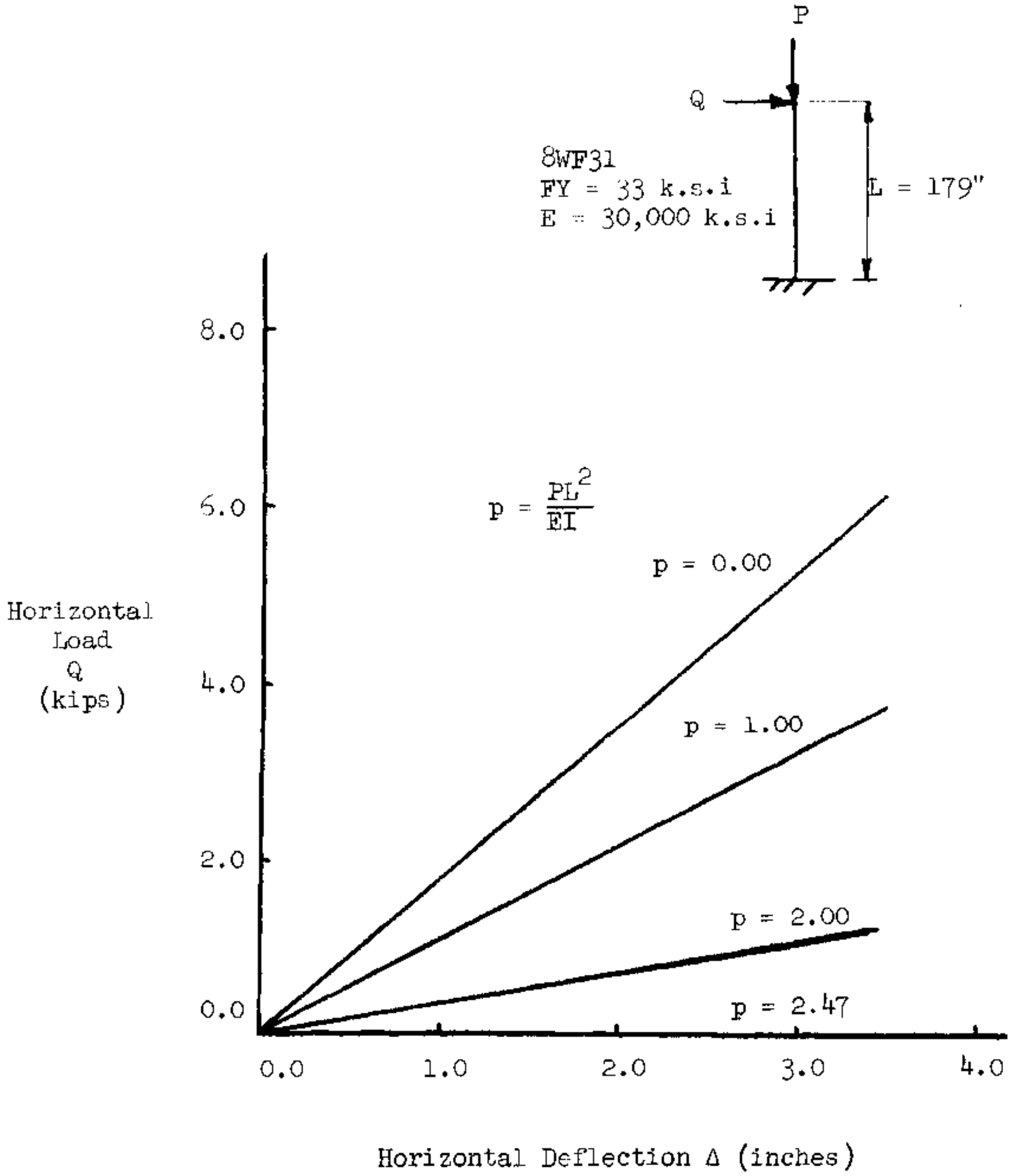


Figure 17. Elastic Load-Deflection Curves for Various Values of the Parameter p for a Cantilever

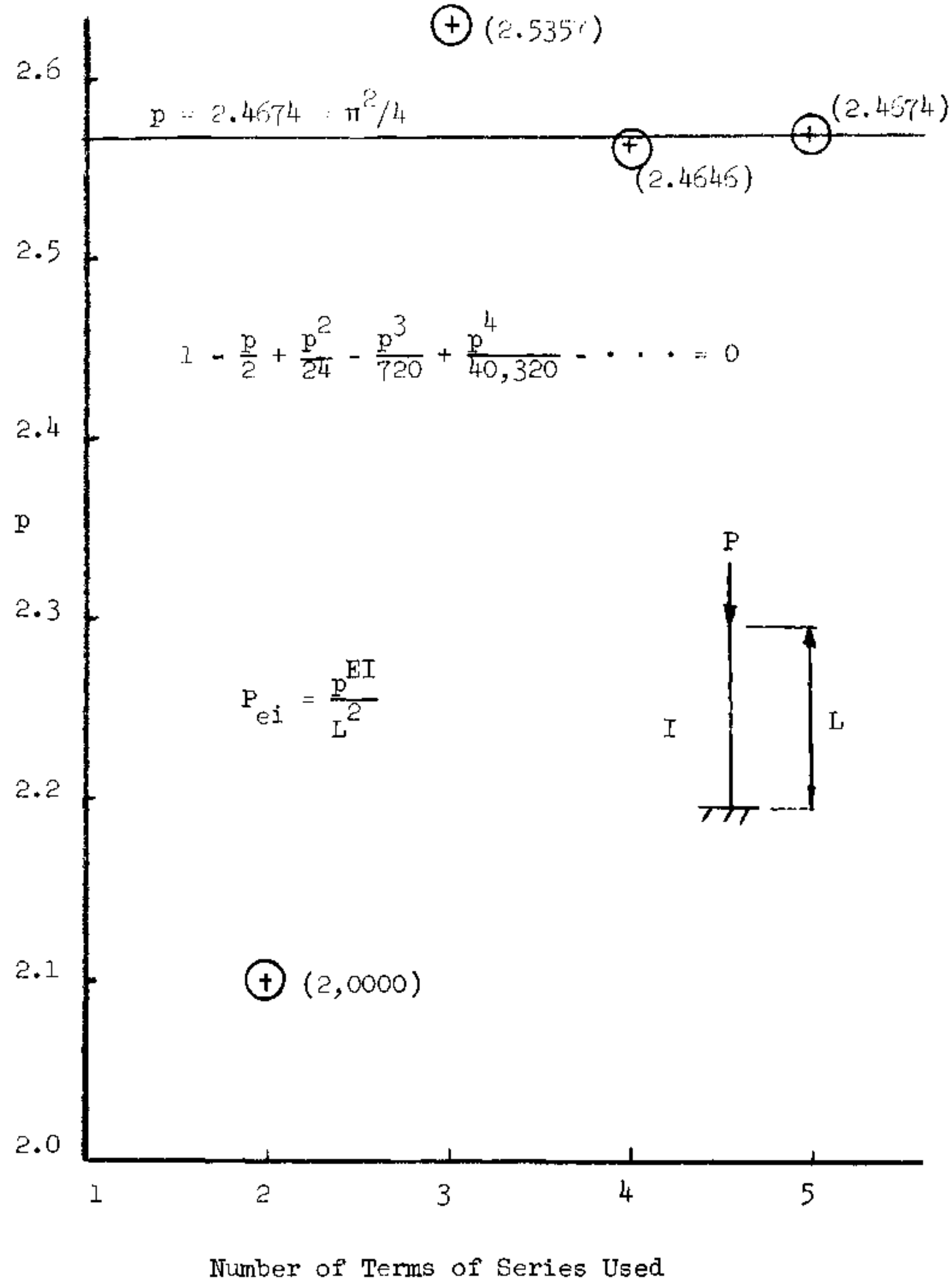


Figure 18. Accuracy of the Elastic Buckling Load as a Function of the number of Terms of Series Used.

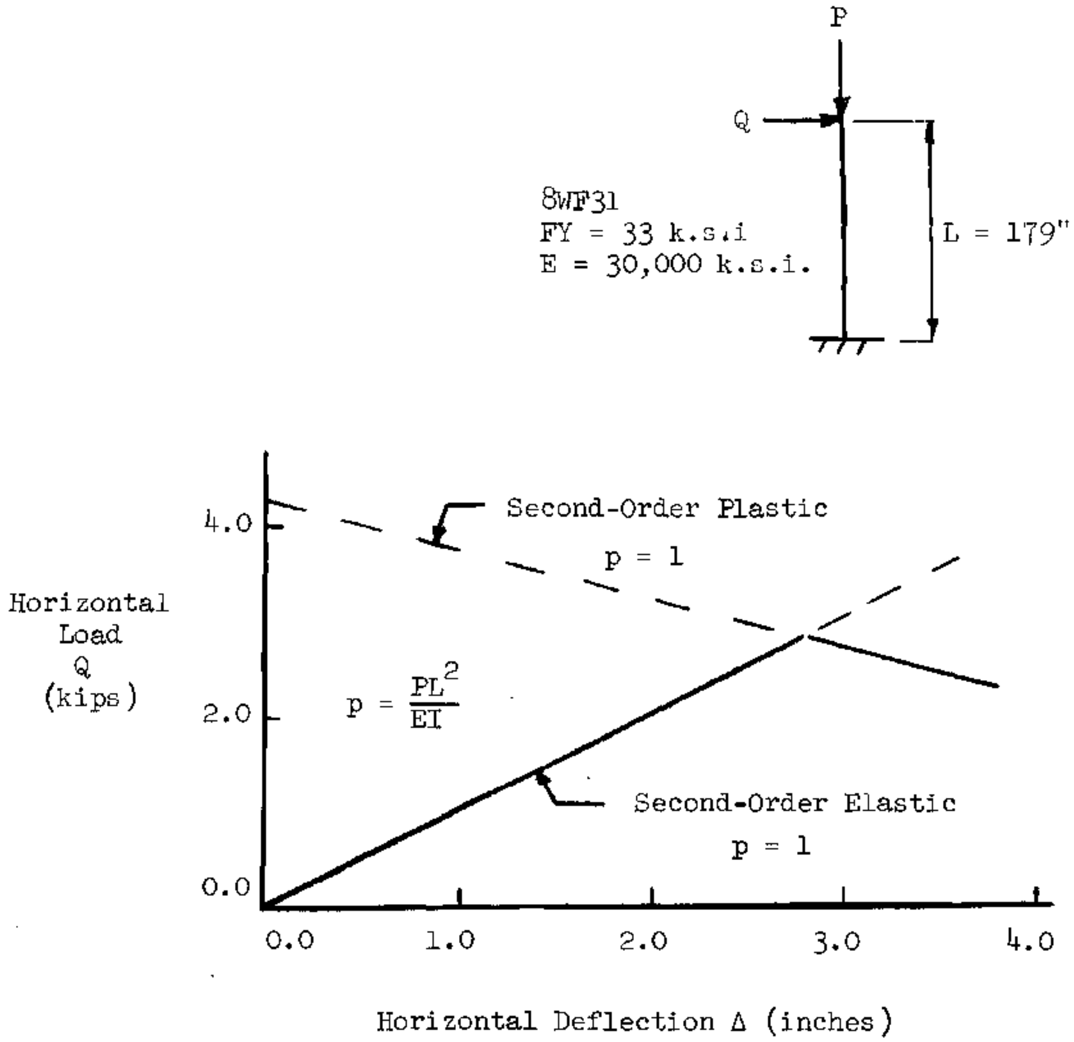


Figure 19. Intersection of Second-Order Elastic and Second-Order Plastic Load-Deflection Curves for a Cantilever

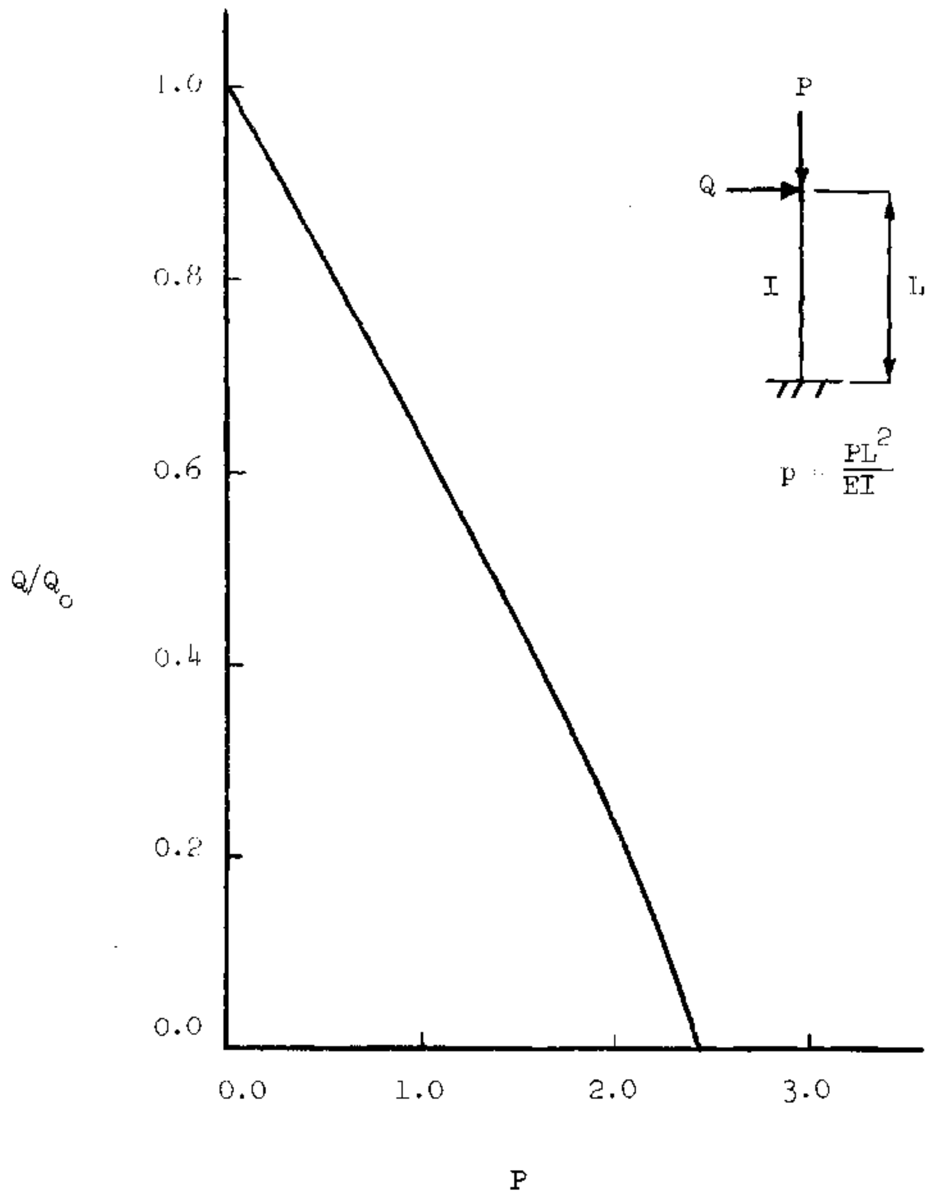


Figure 20. Ratio of Collapse Load by Second-Order Elastic-Plastic Theory to Collaspe Load by Simple Plastic Theory for the Cantilever

(+) Points shown as (+) were furnished by Chu. His reference No. 1 discusses the cantilever but does not give numerical results.

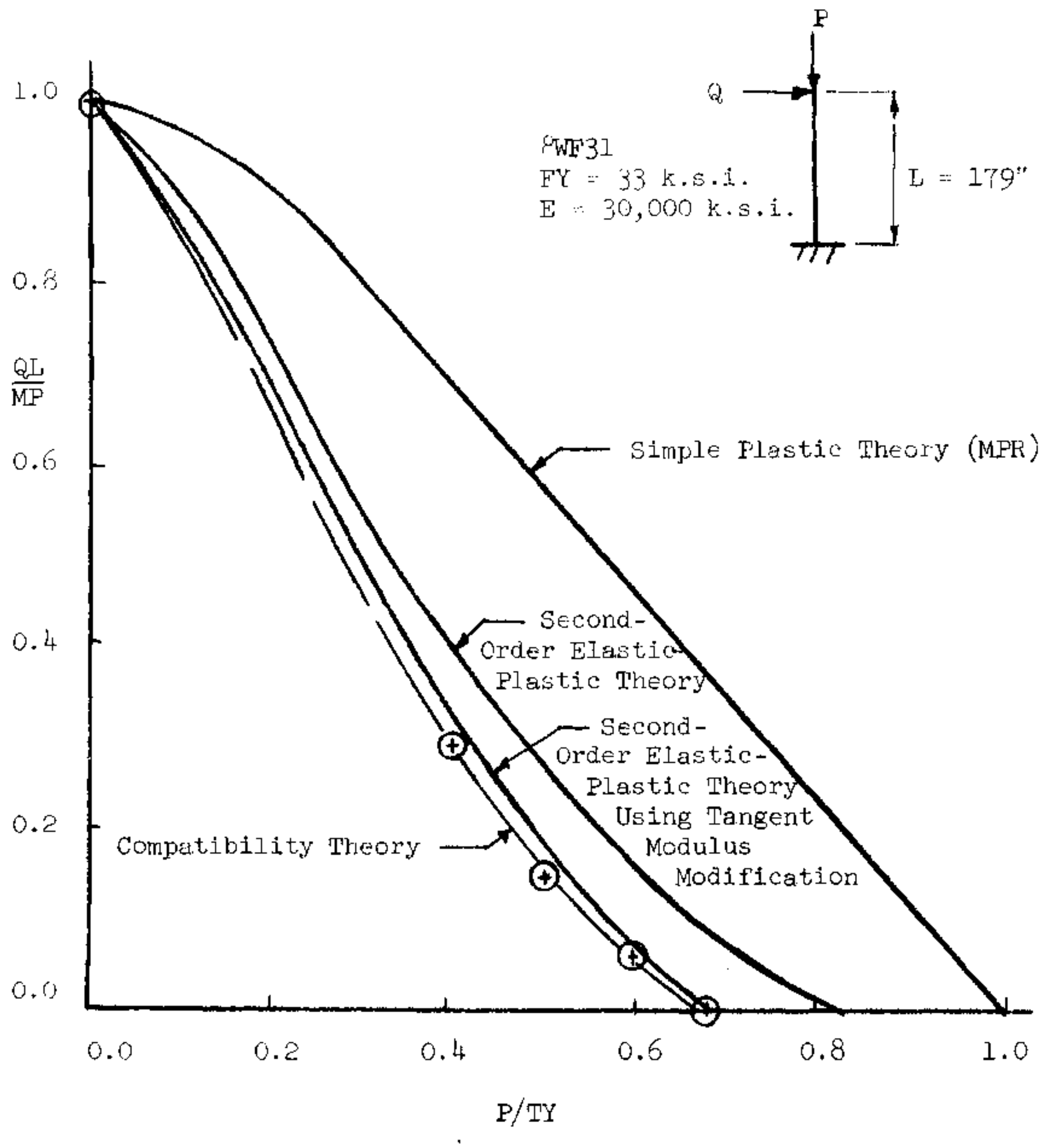


Figure 21. Dimensionless Interaction Curve for a Cantilever by Various Theories

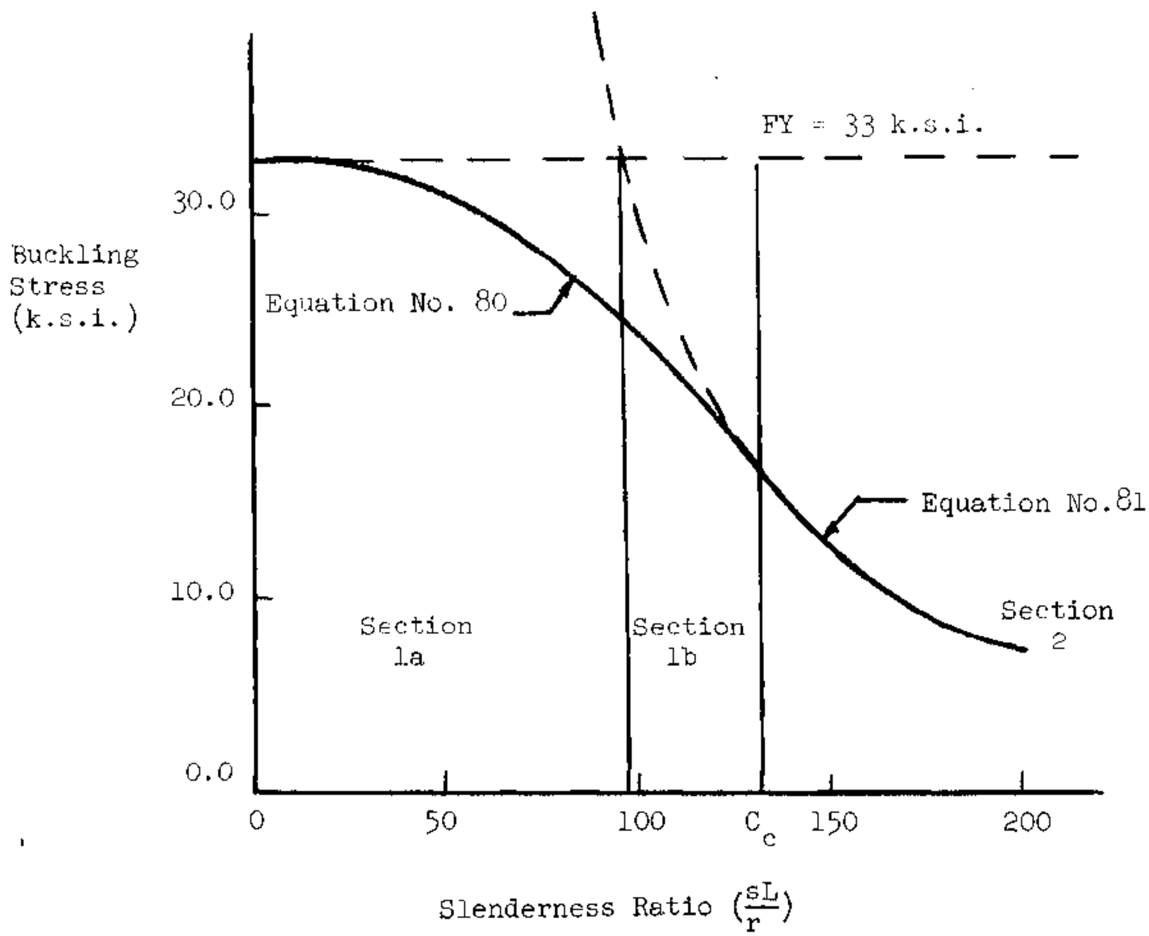
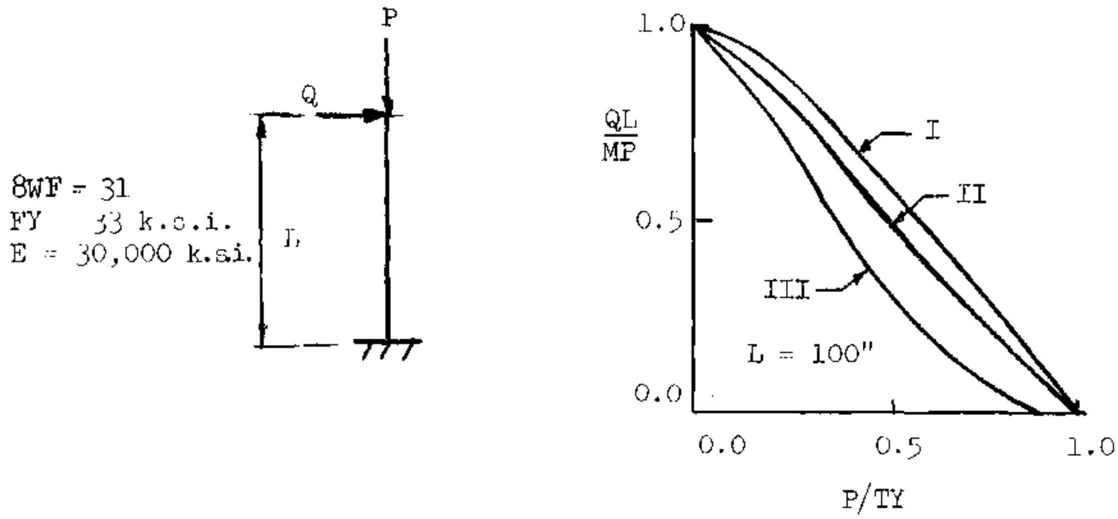


Figure 22. Buckling Stress as Given by the 1963 AISC Manual, for A33 Steel



Curve No.	Theory
I	First-Order Plastic
II	Second-Order Elastic-Plastic
III	Second-Order Elastic-Plastic Using Tangent Modulus

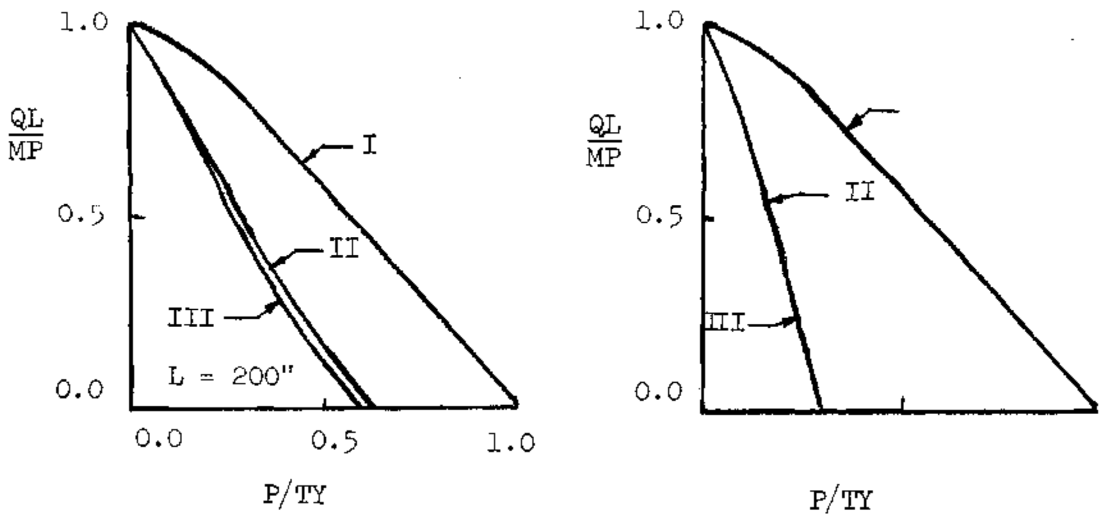


Figure 23. Dimensionless Interaction Curves for Cantilevers of Different Lengths by Various Theories

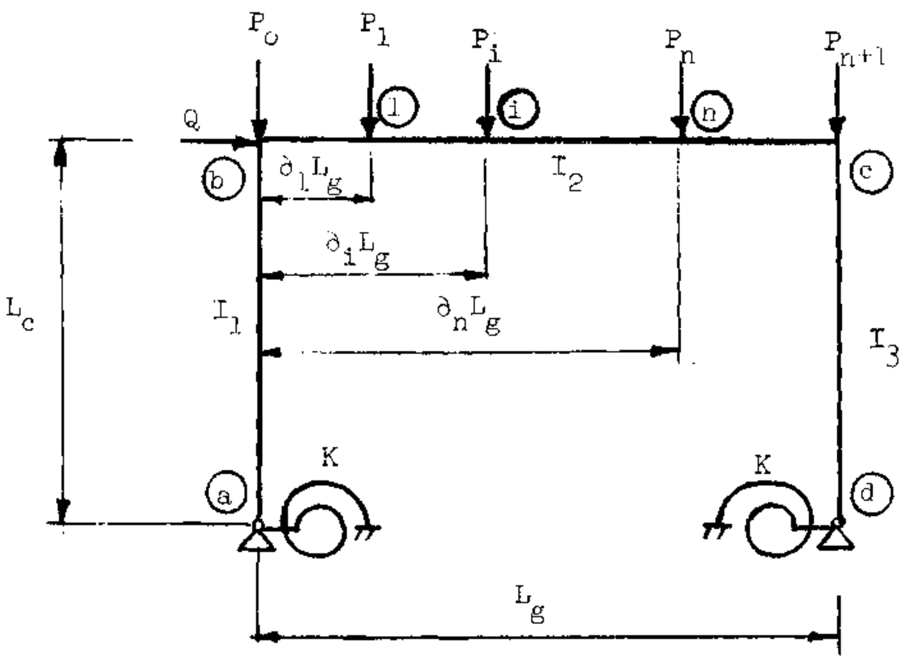
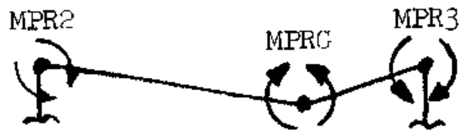
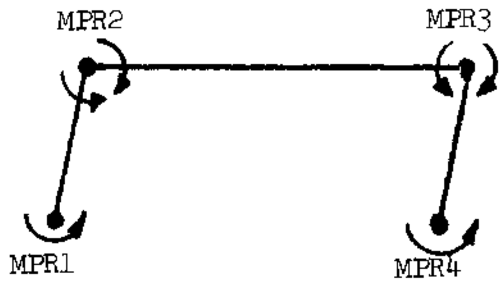


Figure 24. Typical Rigid Bent



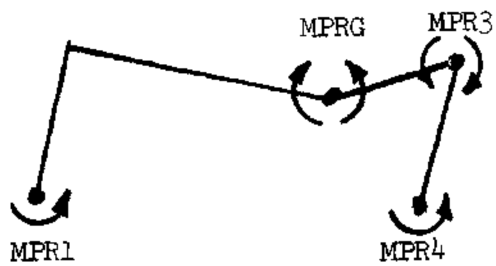
25a

Beam Mechanism



25b

Sway Mechanism



25c

Combination Mechanism

Figure 25. Various Mechanisms for a Simple Bent

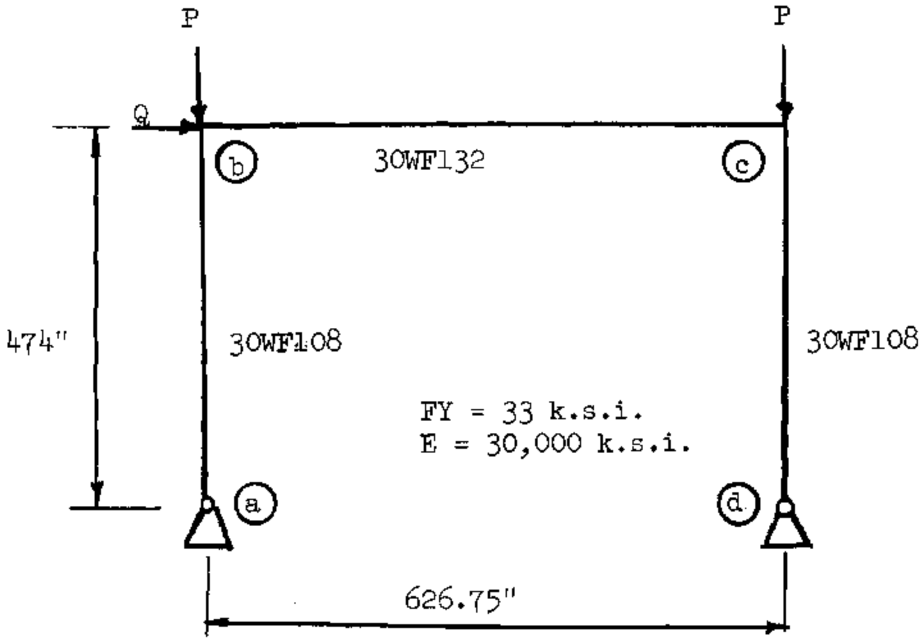


Figure 26. Pinned Base Frame

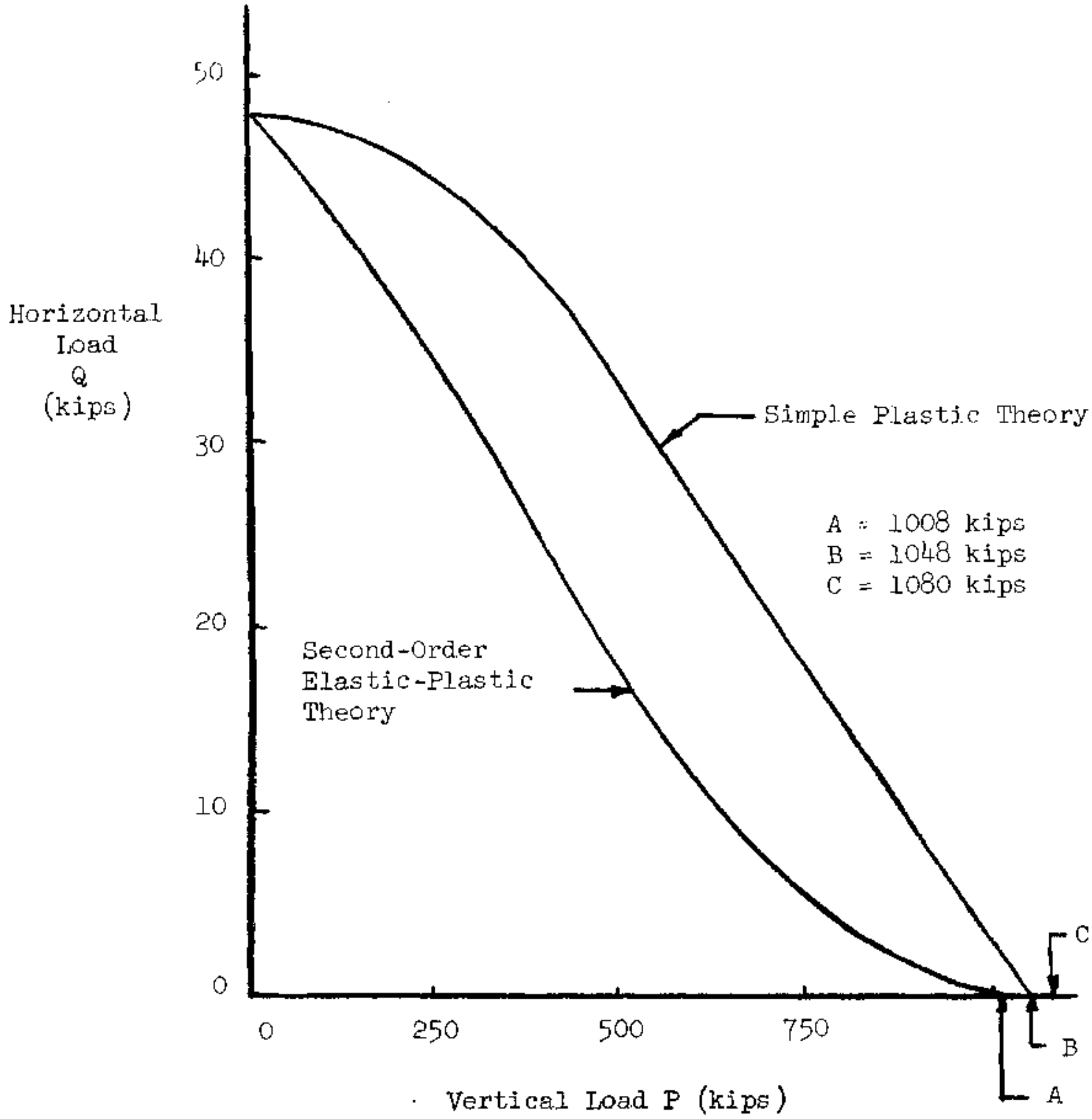


Figure 27. Interaction Curves for the Pinned Base Frame of Figure 26

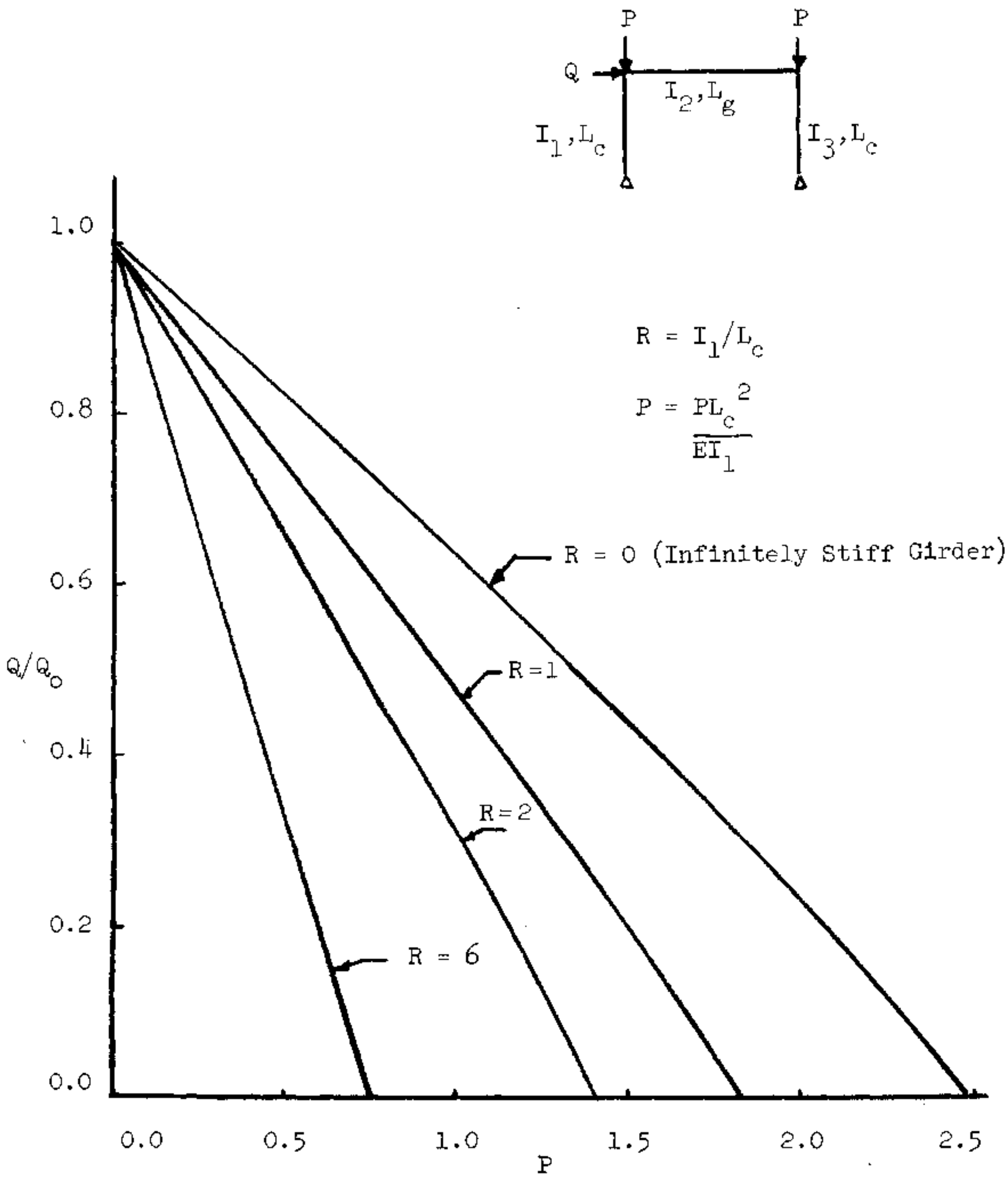


Figure 28. Ratio of Collapse Load by second-Order Elastic-Plastic Theory to Collapse Load by Simple Plastic Theory, for a Pinned Base Frame, Assuming Axial Forces are Equal in the Columns.

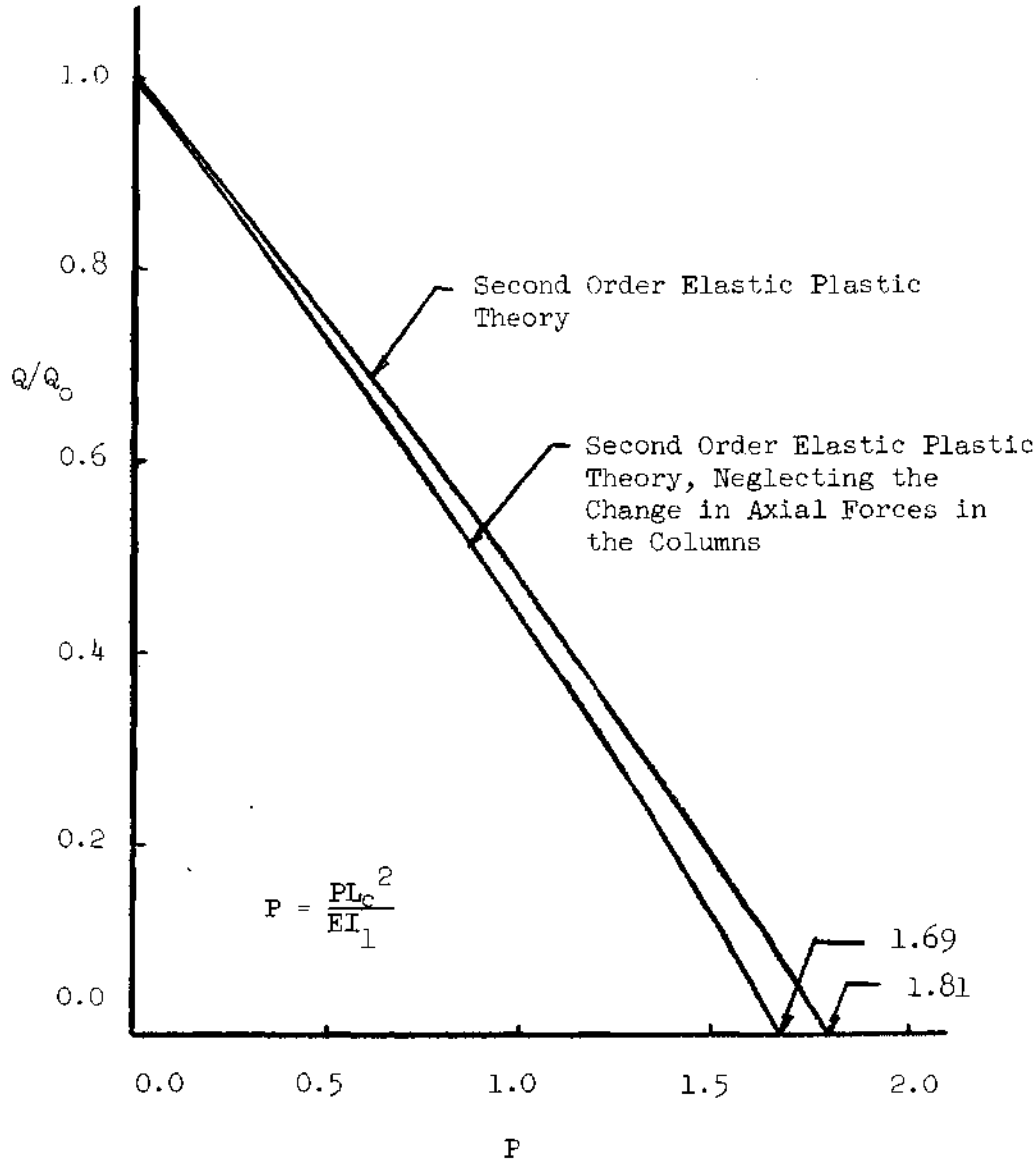
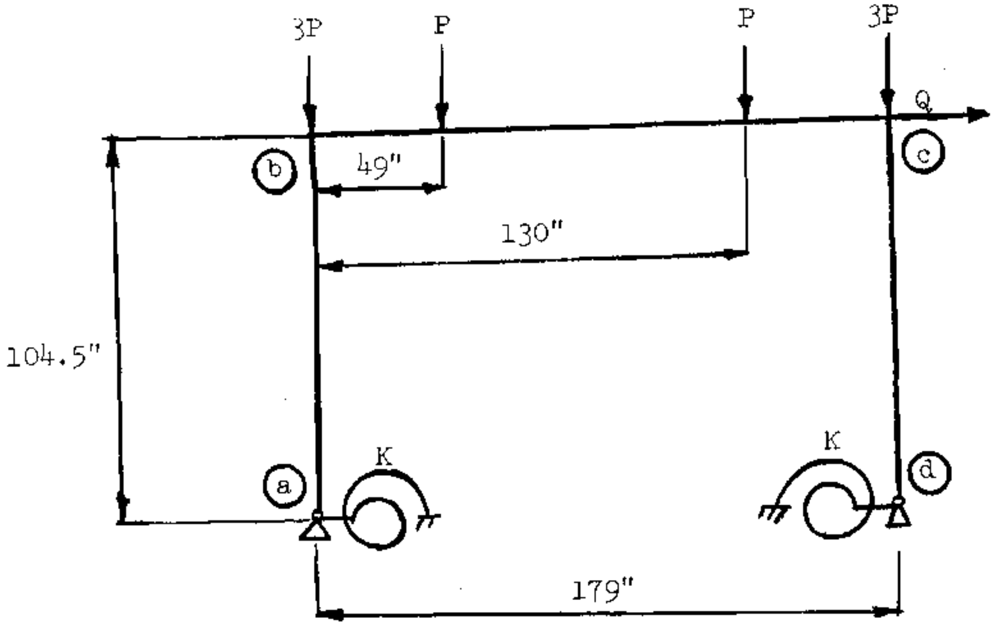


Figure 29. The Effect of Neglecting the Change of the Axial Forces in the Columns on the Interaction Curve for the Frame in Figure 26



P = 20 Kips, K = 133,333 Kip-in/radian

Members	Section	Steel	FY (Coupon Tests) (k.s.i.)	E (k.s.i.)
a-b,c-d	10I25.4	A36	38.57	29,500
b-c	5WF18.5	A441	56.17	29,500

Figure 30. Nominally Fixed Base Frame, Tested at Lehigh (2)

LITERATURE CITED

1. Kuang-Han Chu and Algis Pabarcius, "Elastic and Inelastic Buckling of Portal Frames," Journal of the Engineering Mechanics Division, Proceedings, American Society of Civil Engineers, Vol. 90, No. EM5, October, 1964, p. 221.

2. Peter Adams, P. F. Arnold, and Le-Wu Lu, "Experimental and Analytical Behavior of a Hybrid Frame," Fritz Engineering Laboratory Report No. 297.18, May, 1966.

3. Peter Adams, "Load-Deflection Relationships for Simple Frames," Fritz Engineering Laboratory Report No. 297.18, December, 1964.

4. W. Merchant, et. al, "The Behavior of Unclad Frames," Proceedings of Fiftieth Anniversary Conference of Institution of Structural Engineers, London, October, 1958.

5. S. Timoshenko, Theory of Elastic Stability, McGraw-Hill Book Company, Inc., New York, 1936.

6. G. B. Thomas, Calculus and Analytical Geometry, Addison-Wesley Publishing Co., Reading, Massachusetts, 1953.

7. Joint Committee of the Welding Research Council and the ASCE, Commentary on Plastic Design in Steel, WRC and the ASCE, 1961.

8. American Institute of Steel Construction Committee, Manual of Steel Construction, AISC, New York, New York, Sixth Edition, 1963.

9. Bruce A. Bott, "Elastic-Plastic Analysis of Frames Including Axial Force Effect on Moment Capacity," Fritz Engineering Laboratory Report No. 297.21, June, 1966.

10. M. R. Horne, "Instability and the Plastic Theory of Structures," Transactions of the Engineering Institute of Canada, Vol. 4, No. 2, 1960.

11. George C. Driscoll, et. al, "Lecture Notes," Lehigh University, Department of Civil Engineering, Fritz Engineering Laboratory, Summer, 1965.

12. Niels C. Lind, "Iterative Limit Load Analysis for Tall Frames," Journal of the Structural Division, Proceedings ASCE, Vol. 90, No. ST2, April, 1964.

13. R. L. Ketter, E. L. Kaminstey and L. S. Beedle, "Plastic Deformations of Wide-Flange Beam-Columns," Transactions of the ASCE, 120, p. 1028, 1955.

14. F. B. Hildebrand, Introduction to Numerical Analysis, McGraw-Hill Book Co., Inc., New York, 1956.

15. F. R. Shanley, "Inelastic Column Theory," Journal Aero Science, Vol. 14, No. 5, May, 1947, p. 261.

16. Column Research Council, 1960, Guide to Design Criteria for Metal Compression Members, Column Research Council Engineering Foundation, 1960.

17. M. G. Lay and T. V. Galambos, "The Experimental Behavior of Restrained Columns," WRC Bulletin, No. 110, November, 1960.